

## **Relating university mathematics teaching practices and students' solution processes**

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# Relating University Mathematics Teaching Practices and Students' Solution Processes

PhD Dissertation

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**Relating University Mathematics Teaching Practices and Students' Solution Processes**

PhD Dissertation in Mathematics Education

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*Stine Timmermann Ottesen*  
September 2009

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<sup>1</sup> The hypothesis presented at the end of chapter 4 has been slightly modified. Besides this modification the corrections mainly concern typing errors and grammar.

### Abstract

This dissertation examines if a connection between undergraduate mathematics students' difficulties with task solving in real analysis and the way the students have been taught can be established. The examination is founded on four different empirical studies (preliminary study, pilot study, main study and supplementary study).

The teaching practices are examined through observations structured according to a template developed and adjusted in the preliminary and pilot studies, and tested in the main and supplementary studies. A coarse-grained characterisation of the teaching practice in the main study is partly based on the observation template and a time-line representation (inspired by Schoenfeld) of the teaching, and partly on the analysis of which social and sociomathematical norms and proof schemes (introduced by Harel and Sowder) the teaching practice hinders as well as promotes the establishment and development of.

The students' solution processes are examined through a research design where students in pairs are observed while working on either proof tasks or tasks where the solution includes a proof. The solution processes are partly analysed on the basis of the different stages a solution process can obtain, and partly from a mathematical perspective with a focus on, among other things, the resources of the students, and the interplay between the concept definitions and the concept images the students develop.

From the pilot study a hypothesis is put forward relating the solving difficulties of students to the way the students have been taught. The hypothesis is based on an analysis of the structure and details of a proof and relates how students justify mathematical statements to how proofs are validated in the classroom: *The lack of clarity about what structure and details are in the validation process of a textbook proof in class can contribute to an explanation of the students' difficulties constructing new proofs on their own.* The hypothesis is tested in the main study, and proof validation situations in the teaching practice are thus subjected to a fine-grained analysis.

The analyses of the solution processes reveal several types of complex interacting difficulties. It is difficult for the students to find a proof strategy which could provide them with a proof structure. Instead, the focus is on details with the apparent hope that the proof can manifest itself through a fusion of the details even when the strategy and the structure are not made explicit. Many of the students show signs indicating that a sociomathematical norm of proof production has been established among them. In its most radical form, this norm says that a proof can be constructed just by combining the wordings of several well-chosen theorems selected because they include and combine words appearing in the task. This norm seems related to an established norm that proofs are constructed through the use of tricks. Several of the solution processes are dominated by insufficient mathematical resources, which amplify the confusion over what structure and details are. In several of the processes the students hesitate to look up the formal definitions of the concepts involved, but base instead their reasoning entirely on the concept images they have developed. This is especially the case when the students' concept images do not refer to the formal concept definitions.

The analyses of proof validations in class show that the focus is on the explanation of proof details, whereas the relation between the statement, the proof strategy, and the proof structure receives less attention. Often the students find it difficult to understand the explanation of the details, presumably because the structure is unclear to them. The lack of attention given the connection between the structure and the details can be a factor that sustains the misconceived sociomathematical norm of proof production, while the mathematical connection is hidden. The coarse-grained analysis of the teaching practice combined with an examination of preparation habits also point to possible reasons for the observed difficulties.

## Resumé

I denne afhandling undersøges det om der kan etableres en sammenhæng mellem universitetsstuderendes vanskeligheder med at løse opgaver i reel matematisk analyse og den undervisning de har deltaget i. Undersøgelsen bygger på fire empiriske studier (indledende studie, pilotstudie, hovedstudie samt et supplerende studie).

Undervisningen er undersøgt ved observationer struktureret efter et observations-skema, der udarbejdes på baggrund af det indledende studie, justeres i pilotstudiet, og testes i hovedstudiet samt i det supplerende studie. Den overordnede karakterisering af undervisningen i hovedstudiet tager dels udgangspunkt i observationsskemaet kombineret med en tidslinie-representation af undervisningen (inspireret af Schoenfeld), og dels i hvilke sociale og sociomatematiske normer og bevisskemaer (indført af Harel og Sowder) undervisningen hæmmer og fremmer etableringen og udviklingen af.

De studerende løsningsprocesser er undersøgt ved observationer hvor par af de studerende arbejder sammen om at løse opgaver, der enten er bevisopgaver, hvor der direkte spørges efter et bevis eller opgaver hvor løsningen indebærer et bevis. Løsningsprocesserne analyseres dels ud fra en løsningsproces' forskellige stadier, og dels med et matematisk fokus, hvor de studerendes matematiske ressourcer og samspillet mellem begrebsdefinitionerne og de studerendes begrebsbilleder blandt andet afdækkes.

Pilotstudiet giver anledning til en hypotese omkring sammenhængen mellem de studerendes vanskeligheder med at løse opgaver og den undervisning de har modtaget. Hypotesen tager udgangspunkt i en analyse af strukturen og detaljerne i et bevis: *Den manglende klarhed over hvad der er struktur og detaljer i en bevisgennemgang af et lærebogsbevis i undervisningen kan bidrage til at forklare de studerendes vanskeligheder med at konstruere beviser på egen hånd.* Hypotesen undersøges i hovedstudiet og fokuserer således undersøgelsen af undervisningen til i særlig grad at omhandle situationer hvor lærebogsbeviser gennemgås.

Analysen af studenternes løsningsprocesser afslører flere forskellige typer af vanskeligheder, der interagerer på en ikke trivial måde. De studerende har svært ved at finde en bevisstrategi, som kan lede til en bevisstruktur. I stedet fokuserer de på detaljerne med det tilsyneladende håb at beviset manifesterer sig gennem en sammensstilling af detaljerne, selvom hverken strategien eller strukturen er ekspliciteret. Mange af de studerende viser tegn på at de lader sig styre af en sociomatematisk norm omkring beviskonstruktion, der tilsyneladende ikke før er set rapporteret i litteraturen. I sin yderste konsekvens betyder normen, at en løsning/bevis materialiserer sig selv ved at kombinere ordlyden af et antal (heldigt udvalgte) sætninger. Sætningerne udvælges på baggrund af overfladiske ligheder mellem opgaven og sætningerne. Normen synes at koble til en anden sociomatematisk norm, at beviser konstrueres gennem brug af tricks. Flere af løsningsprocesserne domineres af utilstrækkelige matematiske ressourcer, hvilket forstærker forvirringen omkring og sammenblandingen af bevisstrategien, strukturen og detaljerne. Nogle studerende tøver med at slå den formelle begrebsdefinition op i lærebogen, og bruger kun deres udviklede begrebsbilleder. Det er særlig udbredt i de tilfælde hvor begrebsbillederne refererer til begrebsdefinitioner, der afviger fra de formelle definitioner.

Analysen af bevisgennemgange i undervisningen viser, at fokus er på at forklare bevisdetaljerne, mens der fokuseres mindre eksplicit på sammenhængen mellem sætningen, bevisstrategien og bevisstrukturen. De studerende har ofte svært ved at forstå gennemgangen af detaljerne, fordi bevisstrukturen for dem er uafklaret. Det manglende fokus på sammenhængen mellem strukturen og detaljerne kan være med til at fastholde den sociomatematiske norm omkring beviskonstruktion, fordi de ikke kan se den matematiske sammenhæng. Den generelle analyse af undervisningen samt en undersøgelse af de studerendes forberedelse peger også på mulige årsager til de fundne vanskeligheder.

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# 1 Introduction

“Details are all that matters:  
God dwells there, and you never  
get to see Him if you don’t  
struggle to get them right.”  
(Stephen Jay Gould)

“...lots of things worth say-  
ing can only be said loosely.”  
(William Cooper)

## 1.1 Personal motivation

During my own university study in mathematics I often wondered about the strategies students used when trying to solve mathematical tasks. Even at an advanced level, students seemed to rely on homemade ‘rules’ with no apparent mathematical reference. An example of these rules is given in the following imaginary dialogue between two university undergraduate students majoring in mathematics. The students are trying to construct a proof of a statement given in a textbook task:

**Student 1:** Okay, when we normally solve proof tasks we have to use all the conditions provided in the formulation of the task. So, should we write down the definitions of the conditions in the problem?

**Student 2:** Yes, that’s correct. Let’s do that. There must be an example in the chapter that we can use?

**Student 1:** Yes, that’s a good idea. Or maybe there’s a theorem in the chapter that we have to use?

**Student 1:** (After looking in the textbook) Here we have something that looks the same; oh, this symbol is used slightly differently, but it has to be the same, right?

**Student 2:** Yes, I think you’re right. Let’s try to use this. So we have to change some of the symbols.

**Student 1:** Are you sure that we can just do that?

**Student 2:** Well, I think I also used this theorem in one of the other tasks and the professor said the solution was okay.

Hesitant to talk about the mathematical content, the two students instead focus on strategies not embedded in the mathematical properties of the concepts involved. Students rely on and to some degree practise these superficial strategies instead of trying to learn to use mathematically acceptable forms of argumentation. Rather than solving tasks as a way to learn about mathematical statements and results, mathematical concepts, symbolism, and ways of argumentation, students devote all their attention to decoding ‘the game’. To a certain point, the strategies illustrated in the dialogue seemed to work for many students, but if the professor asked them to solve tasks that required the ability to combine known concepts in new ways, the students would be lost and unable to solve the tasks. It appeared as if they had learned nothing more than superficial strategies that could help them solve only routine proof tasks.

Another ‘insight’ from my own undergraduate days was the (rather obvious) observation that professors teach very differently! Rumours about professors’ different teaching practices could make some students take a course one semester instead of the next if they knew a certain professor was going to teach the course. It was not so much that some professors had a good reputation and some a bad, but rather that their teaching styles differed. Some professors paid attention to technical details and would strive to review all the proofs in the textbook, while others who might not prioritise presenting every proof in the textbook instead spent class time talking about the concepts and making the students solve tasks not in the textbook or gave them assignments such as making concept maps to be discussed in class. My impression from talking to fellow students was that the weak students did not benefit from attending courses where the second type of professors were teaching because they felt they needed the professor to explain the text in detail in order to be able to follow classroom discussions and solve textbook tasks.

Combining these two groups of anecdotal observations, I wondered how university students – also the weak ones – *could best* be taught in order to learn mathematics and not just learn superficial strategies to a degree that would simply allow them to pass the exam. Focusing on mathematical concepts and conceptual relations at the expense of technical details in the textbook proofs did not seem to be without costs, at least for the weaker students. But was that impression true? Is it necessary to train basic proofs skills before moving on to more mature mathematical discussions? Or is it perhaps an advantage also for the weaker students – and maybe the only way to produce professional mathematicians – to push the students toward ‘deeper mathematical understanding’ and minimise routine task solving?

In order to be able to decide which teaching practice is ‘the best’ for specific kinds of students, however, it is necessary to find a way to answer questions like: How can a given teaching practice be characterised? If characteristics have been identified, how do these characteristics connect to students’ learning outcomes? And how do the learning outcomes relate to their problem solving strategies?

## 1.2 Scientific motivation and the formulation of research questions

The first scientific study in the field of mathematics education that I came across was a PhD dissertation written by Johan Lithner [Lithner, 2001].<sup>1</sup> His studies concern undergraduate students' learning difficulties and their ways of reasoning during task solving. Lithner concentrates on students who attend their first calculus course, where proof construction is not the main focus in the tasks the students are trying to solve. Even though the research concerns introductory university mathematics, the findings seem to coincide with my experiences of the way students reason in proof construction situations (as illustrated in the imaginary dialogue).

Lithner identifies three different ways students reason [Lithner, 2003]. He calls the most common reasoning type "*reasoning based on identification of similarities*". In this case, students try to choose examples, rules, definitions, and theorems based on the identification of similarities with the task. The reasoning does not refer to any intrinsic mathematical properties of the concepts involved. Implementation of the problem solving strategy associated with this kind of reasoning consists of copying the procedure from a chosen textbook example. This type of reasoning resembles the behaviour of the students in the imaginary dialogue.

The second type of reasoning is called "*reasoning based on established experiences*" [Lithner, 2000b, p. 168]. Based on prior experiences, the solver chooses solution procedures to base his or her reasoning on, but he or she is aware that the strategy will not guarantee the solution but uses the procedure as an attempt to reach a correct answer. The process does not contain any explicit reference to properties of mathematical components involved in the task.

If, on the other hand, the reasoning is based on properties of the components involved, the reasoning is called "*plausible reasoning*". In his research, Lithner finds that students tend to use reasoning based on identification of similarities even when plausible reasoning might have led them toward a solution [Lithner, 2003]. Lithner hypothesises that the tendency to use superficial reasoning is a result of the teaching environment, since students are not trained to use plausible reasoning [Lithner, 2000b, p. 188]. However, his research does not investigate this assumption, nor does he refer to other studies that could provide support for this hypothesis.

The lack of references to studies that could substantiate this hypothesis strengthened my conviction that an investigation linking problem solving strategies to teaching methods or practices had not, at this point at least, been carried out and was therefore necessary. A further review of the literature did not contradict this impression.

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<sup>1</sup> The dissertation consists of four published papers, and I refer to the publication of these individual papers and not to the actual dissertation when I refer to Lithner's PhD study.

The educational level that Lithner's studies concern does not focus on the understanding of proofs or proof construction. Are his findings applicable to a higher level of university mathematics and can his hypothesis be transferred – and maybe even be verified?

Before I carried out the empirical studies, I formulated an intentional research question based on this mixture of questions and hypotheses:

- **How does teaching practice at university level influence the way students justify mathematical statements in tasks?**

The empirical data material, however, did not leave sufficient opportunities to examine this question thoroughly, because all the justification processes observed, by and large, were dominated by difficulties justifying the mathematical statements introduced in tasks. Therefore, I found it necessary to limit the research question. This resulted in the following two main research questions:

1. **What are the main difficulties university students experience when trying to justify mathematical statements in tasks?**
2. **Are these difficulties related to the teaching practice the students have participated in?**

In order for the reader to understand the research questions, it is necessary to interpret the different keywords used, and to specify the limitations and choices made in order to investigate and hopefully answer the questions.

A *mathematical statement* is a statement that expresses properties of or relationships between mathematical objects or concepts. In a teaching situation, students can be asked to justify different kinds of mathematical statements, for instance, textbook theorems, where they are supposed to reproduce the justification that is the proof of the theorem. I concentrate on mathematical statements given to students as written mathematical tasks, where the proof is unknown to the students. A *mathematical task* is defined as a request that involves mathematics. It is “*a situation in which an individual is presented with an initial set of information and is asked to derive a piece of desired information through the application of permissible mathematical actions and operations*” [Weber, 2005, p. 351]. A mathematical task can be a *problem* in the sense that “*it is not clear to the individual which mathematical actions should be applied*” [Weber, 2005, p. 351-352] to solve it or it can be an *exercise* in the sense that “*it is obvious to the individual which mathematical actions should be applied*” [Weber, 2005, p. 351] to solve it. This definition coincides with Schoenfeld's definition: “*...if one has already access to a solution schema for a mathematical task, that task is an exercise and not a problem*” [Schoenfeld, 1985, p. 74]. Thus, the characterisation of a mathematical task as either a problem or an exercise is relative and depends on the person who tries to solve the task (and also on the formulation of the task). As the point of departure, I am interested in how students act in a

problem solving situation, where the focus is on how they come to realise which actions to apply, and not how they act in an exercise solving situation, where focus necessarily will be on how they carry out the known actions.

To *justify* a mathematical statement means to present mathematical arguments for the truth value of the mathematical statement. Sometimes these arguments come in the shape of a proof. A proof is not a unique entity. It is defined by the context in which it is presented. An acceptable proof of a statement put forward in a second grade classroom is (probably) not acceptable in a mathematics classroom at university. I use the following characterisation of proof, inspired by Stylianides [2007]: A *proof* is a connected sequence of assertions for or against a mathematical statement.

*Teaching practice* is defined as those activities taking place in scheduled periods of time in the presence of a professor, and activities brought about by the students' participation in these scheduled periods of time. Some of the learning activities connected with a mathematics course take place outside the classroom (e.g. preparation activities), and these activities also influence the way students learn to justify mathematical statements. In this study, however, the primary focus is on activities where the professor is present and not on activities taking place outside the classroom.

The remaining word in the research questions that needs explanation is *related*. It goes without saying that all mathematics professors must expect (and hope) that there is a constructive relationship between the teaching practice and how students solve mathematical tasks. But it is also obvious that students cannot learn how to justify mathematical statements and solve tasks without running into difficulties of one kind or the other. Although the professor did not intend these difficulties they might still be a consequence of or related to the teaching practice. But how can this relation be identified? Students are not just products of the current teaching practice but have been influenced by years of experience and participation in various teaching environments. Each of the students has his or her own educational history, prospects and abilities that will cause noticeable differences between their performances. Although the researcher can try to account for the history of each student, it seems impossible ever to be able to verify whether a characteristic feature of a student's solution process is a consequence of the specific teaching practice or a result of prior learning experiences. It is, however, possible to identify important features which characterise the teaching, on the one hand, and the solution processes on the other and make certain relations between the features likely. In some solution processes, one could imagine direct references to the teaching practice, e.g. 'didn't the professor say something about this...?' or 'when we have a task like this don't we normally do ...?', but there is likely to be other implicit signs that could relate solution processes to the teaching practice, e.g. the way students use illustrations, how they solve certain kinds of tasks (e.g. tasks where they have to do an  $\epsilon$ - $\delta$ -proof), what kinds of concept images and concept definition images they have [Tall & Vinner, 1981],



their perceptions of relations between definitions, theorems and proofs and so on.

The research questions state that the study concerns tertiary mathematics education. But they do not specify if different countries, different levels at university or different mathematical topics are involved. As a Dane, I have chosen to primarily focus on the Danish educational system. The educational level is advanced undergraduate and the mathematical topic is mathematical analysis (not calculus). The reason for choosing analysis as opposed to other topics (e.g. geometry or algebra) is that university students traditionally experience major learning difficulties with this topic [Tall & Vinner, 1981; Sierpiska, 1987; Alcock & Simpson, 2001], and it is also here my own personal mathematical interests lie. The reason for choosing to study analysis courses instead of calculus courses is also due to a personal interest in the way students learn to deal with mathematics in a more formal setting and how they learn to construct proofs. As a working definition, calculus concerns the infinitesimal study of *specific* functions, whereas mathematical analysis deals with the infinitesimal study of *general* functions. Since calculus deals with specific functions, calculus courses often put emphasis on operational aspects of concepts (operational as defined in [Sfard, 1991]). Due to the fact that the domain of mathematical analysis is concerned with the study of general functions, analysis courses are more likely to concentrate on structural aspects of concepts (structural as defined in [Sfard, 1991]), and thus to put greater emphasis on proofs and proof construction which makes this topic more appropriate for examining the research questions.

### 1.2.1 The scientific relevance of the research questions

The research question originally intended can be regarded as an example of what Hiebert calls *rich*<sup>2</sup> and *connected*<sup>3</sup> problems in mathematics education:

For illustration purposes, consider the relationship between teaching and learning in mathematics classrooms. That is, *in what ways does the teaching affect learning* and vice versa? How do different instructional approaches lead to different kinds of learning? These questions define a problem that is rich. It is nontrivial and multifaceted. Pursuing a solution to the problem has triggered numerous additional questions that have received attention from a variety of perspectives... [Hiebert, 1998, p. 143] (italics have been added).

Hiebert advocates the view that the problems studied in mathematics education research should always be rich and connected. The difference between the formulation of the research question originally intended and Hiebert's formulation of relevant problems in mathematics education is first of all that Hiebert

<sup>2</sup> "*Rich problems...can be approached from a variety of perspectives, and the process of solving them is often filled with intermediate results and insights*". [Hiebert, 1998, p. 142]

<sup>3</sup> Connected problems are problems that in some way are related to other problems. [Hiebert, 1998, p. 143]

talks about how teaching affects learning in a broad sense, whereas learning in both the question intended and in the first research question is restricted to solution processes. Learning could also be analysed through, for instance, students' mathematical activities in the classroom or their written responses to assignments or other sorts of written questions, as is the case, for instance, in [Cobb et al., 2001] and [Stephan & Rasmussen, 2002] or in [Dreyfus, 1999], [Selden & Selden, 1995] and [Durand-Guerrier, 2003], respectively. Second, Hiebert also looks at how learning affects teaching, an issue not considered in my research question.

In order to answer the research questions, it is necessary to find ways to characterise teaching practices. A teaching environment is, by nature, a social construct with the professor and the students as (more or less) active participants. The research questions suggest that the characterisation in some way should be connected to a characterisation of the students' solution processes. But the solution processes are necessarily individual and psychological in nature. Consequently, a study of the research questions requires the use of different perspectives and ways to approach the problem exactly as Hiebert advocates.

A lot of studies concerning university students' ways of solving mathematical tasks have been carried out over the last ten years (work done by, for instance, Lithner [2003], Weber [2005], Raman [2003], and Durand-Guerrier & Arsac [2005]). I have only come across one study that specifically tries to characterise a teaching practice at university level [Weber, 2004], but several characterisations of teaching practices exist where data from the primary and secondary level have been used in the development, for instance, [Cobb et al., 1997], [Schoenfeld, 2000] and [Bass & Ball, 2004]. A study of the research questions is thus connected to previously addressed problems in the community of mathematics education research and is in that respect "*a connected problem*". The proposed research questions are thus relevant and worth investigating.

### 1.2.2 How might this study contribute to the body of mathematics education research?

The research questions are examples of the fundamental question of how teaching and learning (and learning difficulties) are related. In order to investigate this, however, it is necessary to narrow down the area of research. The questions focus on a specific educational level, the tertiary level, which during the last decade has received increasingly more attention in the mathematics education research community. Furthermore, the questions have been restricted to a specific topic, real analysis, which many studies have been concerned with because of students' difficulties in learning the concept of limit. And, last, only learning as it appears in task solution processes is considered and not all kinds of learning.

This dissertation contributes with a study of how university students are being taught and how teaching influences their learning opportunities. The research provides a way to characterise teaching practices and to link this characterisation to students' solution processes. How the findings more concretely contribute will

be discussed later in the dissertation after the results have been presented.

As an added bonus, the links found between solution processes and teaching practices gave rise to some didactical considerations and suggestions concerning the way textbook proofs are demonstrated in class and how students might learn to improve their preparation.

### 1.3 Who might benefit from reading this dissertation?

The mathematical content in this dissertation is undergraduate real analysis with strong emphasis on proofs. Real analysis at this level is based on properties of the real number domain, number sequences, limits and convergence, continuity, differentiability and integrability in a formal setting. The formal setting is that of the Weierstrassian analysis, where limits and thus continuity, convergence, differentiability and integrability are formulated using the  $\epsilon$ - $\delta$ -formalism. In this formalism a real function  $f$  defined on a nonempty subset  $E$  of  $\mathbb{R}$ ,  $f : E \rightarrow \mathbb{R}$ , is continuous at a point  $a$  if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $|x - a| < \delta$  implies  $|f(x) - f(a)| < \epsilon$  for  $x \in E$ .

In order to understand the mathematical content and the discussions related to mathematics in this dissertation, it is advisable that the reader is acquainted with the terminology, how relevant concepts are defined using this formalism, and how proofs based on this formalism are composed.

The nature of the PhD project is descriptive and analytical. It does not include any developmental collaboration with professors or any aspects of design research. Even though the primary focus, however, has not directly been to investigate measures meant to improve teaching and learning, a concrete tool for teaching proofs at the university level has in fact come out of the research project as a by-product. The analyses and the findings are thus not only relevant for the research community of mathematics educators, but also for university students and professors teaching at this level or higher levels who might benefit from reading the dissertation.

### 1.4 Overview of the research design

Different countries and scientific communities have different traditions when it comes to the content of a PhD dissertation. Some use the format of articles and attach an introduction binding the different papers together. Others have traditions for writing monographs, where the PhD study is presented as one unified report.

I have chosen to present my work in the monographical form partly because of tradition and partly because I find this form more suitable for a thorough and coherent presentation and discussion of my research, since it has been carried out as *one* three-year-long project. This form of presentation allows more room for a methodological discussion and the presentation of longer pieces of empirical

data for the data analysis. These aspects contribute to ensuring the reliability of the approach. Concurrently with writing the PhD dissertation, some of the results have been published in a Danish peer-reviewed journal ([Timmermann, 2007a]) and in the proceedings of the Fifth Congress of the European Society for Research in Mathematics Education ([Timmermann, 2007b]).

Reporting a three-year long project in a linear way, as is necessarily the case with a written text, requires making some choices. One might choose to present the study chronologically to emphasise the process and the rationale behind many of the choices made during the study. This form of presentation runs the risk of being very long, and while the conclusions first appear at the end (since this is where they chronologically belong), the reader might be kept in ignorance throughout the entire dissertation and have trouble locating ‘the connecting thread’. Another disadvantage of the chronological form is that all the steps in the process are not necessarily scientifically important.

Another choice could be to focus strictly on the research questions and evade all parts of the project not directly involved in the answers. This might exclude intermediate steps in the research process, and some of the research choices could thus be seen ‘as taken out of the blue’. I intend to mix the two approaches in this dissertation. This means that the dissertation will contain a description of the choices I have made, and focus on more than just the answers to the research questions. For this reason, I present an overview of the research process in this chapter and describe the aim and use of the different empirical studies and how they are linked. The dissertation contains a methodological chapter (Chapter 3), where the particular parts of the research design will be discussed in greater detail.

The empirical data material is divided into four parts. A preliminary study followed by a pilot study and a main study, and, finally, a supplementary study. The research design had not been formulated in the beginning of the project, but developed during the study as my experience, insight into the problem area, and scientific focus grew and sharpened.

#### 1.4.1 Preliminary study

The preliminary study was carried out in autumn 2004. I visited three different Danish universities, where I observed an undergraduate mathematics course at each of them. The purpose was partly to see how teaching at this level was conducted at different universities, and partly to investigate what kinds of questions I could answer by observing teaching practices and students’ solution processes.

The first course, course A, was an advanced analysis course in the sense that the notions of continuity, convergence, function spaces and integrability (Riemann-Stieltjes integration) were studied in the abstract frame of metric spaces. At this university, understanding textbook proofs and construction of proofs was the main focus. All students majoring in mathematics or physics or a subject related to these two subjects (actuarial science, economics, statistics,

geophysics, etc) were required to take this course right after an introductory calculus course. The course was divided between lectures (two hours) for all students held by the professor in charge of the course, and tutorials<sup>4</sup> (three hours) for smaller groups of students, where teaching assistants were responsible reviewing task solutions and for helping students solve assigned tasks. The textbook ‘Real Analysis’ [Carothers, 2000] was used. I observed two lectures and four tutorials.

The second course, course B, was for math/tech engineers<sup>5</sup>. The content of the course was a mix between linear algebra and linear differential equations of one and several variables, and emphasis was put on the use of computer tools for calculatory purposes and for conceptual understanding, and on applicational aspects. Proofs and learning to construct proofs was not emphasised. The course was divided between auditorium lectures for all students (one hour) and solving sessions (three hours) for smaller groups of students where the students solved tasks in groups with the assistance of the professor/teaching assistants. The Danish textbook ‘Matematisk analyse 1 [Mathematical Analysis 1]’ [Jensen et al., 2000] was used. Two lectures and two tutorials were observed.

The content of the third course, course C, was continuity, differentiability and integrability (Riemann integral) of one variable functions. The students who attended the course were all majoring in mathematics and this was their first analysis course after an introductory calculus course. Focus was on proofs, both understanding textbook proofs and construction of proofs. The course was divided between lectures held by the professor, and solving sessions where the students solved tasks individually or in groups with the professor available to offer help. The course used the textbook ‘An Introduction to Analysis’ [Wade, 2004]. Four lessons<sup>6</sup> (three hours) were observed.

Based on the preliminary study of different teaching practices a course was chosen for a more thorough study planned for the spring of 2005. The universities where courses A and B were given were not considered for different reasons: In course A, the professor responsible for the course planned and structured the content of the lectures as well as the tutorials, and the teaching assistants were thus constrained and not free to act according to their own views on teaching. This construction would give rise to several teaching practices in the same course, which causes methodological complications (for instance, when interviewing the teaching assistants they would have to speak on behalf of the professor in charge of the course). It was my impression that all undergraduate courses at this university were structured in this way, which is why this university was not selected

<sup>4</sup> I separate between *tutorials* where a professor or a teaching assistant primarily spend time reviewing solutions to tasks, and *solving sessions* where students primarily spend time solving tasks with the assistance from the professor/teaching assistant.

<sup>5</sup> Students who had chosen one of the following areas of study: Electro-technology, communication technology, mathematics & technology, and software technology.

<sup>6</sup> In this course, the lectures and the solving sessions are not scheduled to two separate time slots, but instead combined into a three hour lesson.

for further research.

In course B the main focus had not been on proofs. Construction of formal proofs first appeared at a much later stage in the educational programme at this university. So it would have been necessary to study a higher level course if the course looked at were to include proofs as a major part of it.

In courses A and B the lectures were dominated by the professor. In both courses, the professors only asked very few questions, and when they did, the students who volunteered to answer always answered correctly. From auditorium observations, it was impossible to draw any conclusions regarding students' perception of the teaching practice. In course C, on the other hand, the lectures took place in a classroom with a smaller number of students which made dialogue and remarks from the students possible. This was the case for many of the courses given at this university.

### 1.4.2 Pilot study

In the light of these considerations, a course at the university where course C was given was chosen and a pilot study was prepared. The pilot study was meant to result in the development of a way to characterise teaching practices. The course chosen was a continuation of course C, so the professor and the students remained the same, but the content of the course was now more advanced and included continuity, differentiability and integrability (Lebesgue integral) for functions of several variables in the abstract frame of metric spaces. The course, which had the same structure as course C, also used the textbook 'An Introduction to Analysis' [Wade, 2004], but only for the first two months. During the rest of the course the students used notes on measure and integration theory written by the professor. Fourteen students took the course. The pilot study resulted in three categories of empirical data:

- Classroom observations of each lesson (with a few exceptions due to other commitments). The lectures were video-taped or audio-taped.
- Observations of students in constructed task-solving situations, where students tried to solve tasks in pairs without help from the professor. The tasks were devised by me. The situations were audio-taped and transcribed.
- Two sets of individual interviews with the professor and the students. The interviews were audio-taped and transcribed.

In the first group of data, a preliminary observation template containing categories for characterisation of the teaching practice constituted an underlying basis for the classroom observations. In the second group of data, the students were not interviewed during or after the task-solving sessions which is why I do not refer to them as 'task-based interviews' – an otherwise commonly used data construction method in mathematics education research (see e.g.: [Raman, 2003; Lithner, 2003; Weber, 2005]). In the third group of data, the individual interviews were carried out mid-way through the course and after the final exam.

For several reasons, the data material was inappropriate for it to function as the sole source of empirical data for studying the research questions. First of all, the course being studied was a continuation course, so social and socio-mathematical rules and norms (as defined by Cobb et al. [1997]) had already been established in the previous course. This meant that the students, to some extent, already knew what was expected of them when constructing new proofs, and they had already been introduced to the formal setting of analysis. Second, the content of the course was rather advanced, and it turned out that many of the difficulties the students experienced were caused by a lack of understanding at a lower level. Third, a research interest in how a dialogical teaching practice could be established arose based on the pilot study. In order to investigate the conditions for establishing a teaching practice based on dialogue, I found it necessary to conduct a more thorough investigation of the students' study habits (in the pilot study I had only *asked* the students about their study habits) in order to draw substantial conclusions. And fourth, the students' body language and periods of silence concealed too much information. This necessitated videotaped recordings.

### 1.4.3 The main study

Based on my experiences with the pilot study a main study was designed and carried out in the autumn of 2005. The same professor taught the course, which had the same content as course C in the preliminary study, but with a new group of students. Twenty-four students attended the course. The class met twice a week for three hours over a period of fifteen weeks. The course ended with a final exam with pass/fail marks. The course used the textbook 'An Introduction to Analysis' [Wade, 2004].

The design of the main study resembled the pilot study in many ways, but some changes and extensions were made. The observation template was adjusted as a consequence of the pilot study. New categories were included and others had been divided up. Selected groups of students were observed while they tried to solve textbook tasks in the classroom, and an additional study of the students' preparation habits was also carried out. Ten students volunteered to record their study habits before each lesson that specified the amount of time spent preparing and how that time was divided between different types of preparation (reading, solving tasks, reviewing proofs, etc.). To summarise, the main study contained the following categories of empirical data:

- Classroom observations of each lesson (with a few exceptions due to other commitments). The lectures were video-taped or audio-taped.
- Observations of students in task-solving situation in the classroom. These situations were video- or audio-taped.
- Observations of students in constructed task-solving situations, where students tried to solve tasks in pairs without help from the professor. The

situations were video-taped and transcribed.

- Investigation of students' preparation habits.
- One set of individual interviews mid-way through the course with the students who participated in the preparation survey. An interview with the professor. The interviews were audio-taped and transcribed.

#### 1.4.4 Supplementary study

After the completion of the main study I had the opportunity to observe an analysis course at a Canadian university in the spring of 2006. The data from this study and from the preliminary study are used in order to compare and contrast the teaching practice observed in the main study with teaching conducted elsewhere. Even though the course in the main study was not an experimental course in the context of the specific university, the course organisation and the goals of the professor differed from the other courses I had observed during the whole project; in order to discuss the generalisability of the findings it was necessary to compare the course in the main study with courses with similar content offered at other universities.

The course in the supplementary study was a bridge course between calculus and rigorous mathematical analysis, and the content of the course resembled the content of course C (continuity, differentiability and integrability (Riemann integral) of one variable functions, but included also proof techniques, topology of the reals, and compact sets. On average, 35 students attended the lectures. The students who took the course were considering majoring in mathematics, but had not yet made a final decision. The course was divided between lectures (50 minutes) for all students held by the professor in charge of the course, and tutorials (50 minutes) for smaller groups of students where a teaching assistant reviewed task solutions. The course used the textbook 'Analysis with An Introduction to Proof' [Lay, 1990]. Fourteen lectures and two tutorials were observed.

#### 1.4.5 Overview of the research process

Study	Aim	Use
<b>Preliminary</b> <ul style="list-style-type: none"> <li>• Observations of lectures, tutorials and solution processes at three Danish universities</li> </ul>	<ul style="list-style-type: none"> <li>• Investigating types of answerable research questions</li> </ul>	<ul style="list-style-type: none"> <li>• Developing ideas for researchable questions</li> <li>• Selecting a course for further studies</li> <li>• Data for comparing and contrasting</li> </ul>



<b>Pilot</b> <ul style="list-style-type: none"> <li>• Classroom observations</li> <li>• Solution processes</li> <li>• Interviews</li> </ul>	<ul style="list-style-type: none"> <li>• Developing an observation template</li> <li>• Investigating relations between teaching and solving processes</li> </ul>	<ul style="list-style-type: none"> <li>• Background information</li> <li>• Hypothesis development</li> <li>• Students' solving difficulties</li> </ul>
<b>Main</b> <ul style="list-style-type: none"> <li>• Classroom observations</li> <li>• Solving observations</li> <li>• Solution processes</li> <li>• Interviews</li> <li>• Preparation logs</li> </ul>	<ul style="list-style-type: none"> <li>• Investigating classroom dialogue and preparation habits</li> </ul>	<ul style="list-style-type: none"> <li>• Empirical study of the hypothesis</li> <li>• Students' solving difficulties</li> </ul>
<b>Supplementary</b> <ul style="list-style-type: none"> <li>• Observations of lectures and tutorials at a Canadian university</li> <li>• Solution processes</li> <li>• Interviews</li> </ul>	<ul style="list-style-type: none"> <li>• Exploration of other teaching practices</li> </ul>	<ul style="list-style-type: none"> <li>• Data for comparing and contrasting teaching practice characterisations</li> </ul>

## 1.5 Structure of the dissertation

This first chapter in the dissertation contains the motivation, both personal and scientific for the research questions, together with an overview of the research process. The following chapter, **chapter 2**, is a literature survey of the topics within mathematics education related to the research questions. The review is also meant as an introduction to different theoretical constructs that I use in the data analysis.

**Chapter 3** contains the methodology of the PhD project. There seems to be no agreement in practice about what a methodological chapter should contain. In many papers the section termed 'methodology' often presents nothing more than a short summary of the methods applied in the respective study. I use methodology as an umbrella term for the design of the investigation, the way data have been analysed, and the analytical reflections concerning the consequences of the choice of methods. Since some design choices were made on account of data analysis of specific parts of the design, the methodological chapter contains conclusions not yet accounted for. Consequently, the reader is asked to have faith

in the accuracy of the conclusions at this point (one option would be to skip the methodological chapter and proceed to the data analysis and discussion).

Based on the pilot study, a hypothesis concerning students' difficulties in constructing proofs is put forward. **Chapter 4** gives an account of this hypothesis and how it emerged from the data. Next, **chapter 5** presents the characterisation of the teaching practice observed in the main study, compares the characterisation with data from the supplementary study, and includes the examination of the part of the proposed hypothesis which concerns the teaching practice. **Chapter 6** contains the analysis of constructed solving sessions with the main aim to test the part of the hypothesis which concerns students' solving difficulties. The chapter ends with a summary of the other types of difficulties found in the processes. In the last chapter, **chapter 7**, the research findings are summarised and discussed. This chapter also contains a discussion about how the research design influences the results obtained. I end with some reflections regarding the didactic pedagogical usefulness of the work presented in this dissertation.

The **appendices** contain information about the course examined in the main study, the interview questions, the tasks used in the different studies, the preparation log, graphical representations of data analyses, and a list of definitions and theorems referred to in the dissertation.

## 1.6 Language, transcripts of data, and rules of transcript

The language of this dissertation is English, but almost all interviews, solving sessions and teaching took place in Danish (except the supplementary study conducted in Canada). The interviews and the solving sessions have all been transcribed (in Danish), but only selected teaching episodes have been transcribed (in Danish). These transcripts are all appended. It has, however, not been beneficial or affordable to translate all the transcripts into English. Transcript excerpts that have been included in the main text have been translated into English. I have attempted to translate the transcripts into fluent English (spoken language) instead of a word-for-word translation. In order to do that, it has been necessary in some places to make use of interpretations of the meaning of the spoken words.

The students' names have been replaced with pseudonyms in the transcripts. Throughout the dissertation each student is given a single alias that matches the student's gender. The students participating in the research on solution process have been giving pseudonyms where the first letter corresponds with the name of the particular team (e.g. Bill is on team B). I use my initials 'STO' to indicate myself. The professor in the pilot study and in the main study is referred to as Michael or the professor.

In transcripts of interviews, solving episodes, and teaching episodes '.' indicates that the person speaking interrupts him- or herself or has been interrupted by another person, while '...' indicates a break where no one is speaking. A response from a person described by 'hmm' means that the person is reluctant to

agree with what has been said, while ‘mmm’ indicates that the person confirms or agrees with what has been said. Short interruptions or confirmations by people other than the speaker will be placed in square brackets, while my comments on actions such as where a person is looking or what he or she is writing are placed in brackets.

## 1.7 References and quotes

I use three different ways to refer to literature, either in the form of [Sfard, 1991], [Sfard, 1991, p. 20] or Sfard [1991]. Some examples will illustrate the differences: ‘Sfard [1991] discusses the dual nature of mathematical concepts and she argues that there is a “*deep ontological gap between operational and structural conceptions*” [Sfard, 1991, p. 4]. The dual nature of mathematical concepts has also been studied in [Gray & Tall, 1994]’. The notation ‘ibid.’ is not used in this dissertation.

Quotes (from written texts) are written with double quotation marks and in italics. Longer quotes are indented and in a smaller font size. In quotes, (...) indicates a break of one or more sentences within the quotation, while ... means that one or several words in the sentence have been omitted.

## 2 Placing the study in the scientific landscape

“La mathématique est l’art de donner  
le même nom à des choses différentes.”  
(Henri Poincaré)

“There is a risk that the discipline of  
didactics of mathematics will become  
*“L’art de donner, aux mêmes choses, des  
noms différents”.*” (Anna Sierpinska)

The previous chapter provided a framework for this study. In addition, the personal and the scientific motivation behind the formulation of the original research question and the two final research questions were presented as well as argumentation concerning the relevance of the research questions.

The purpose of this chapter, which places the study within the context of relevant research literature, is to present what has already been said and executed by other researchers on issues related to the research questions, and to present the scientific results relevant for the analysis of my data. The latter of the two aims means that the theoretical constructs and frameworks I use in the data analysis are presented in greater detail than the ones I do not apply.

In order to structure the literature survey, five main themes have been extracted from the research questions:

1. Learning mathematical analysis.
2. Acquisition of the role of the justification of mathematical statements in mathematical analysis.
3. Teaching mathematical analysis.
4. Justifying mathematical statements in mathematical analysis.
5. Student problem solving strategies in mathematical analysis.

The rationale behind the order of the five themes is as follows. When talking about teaching and learning of mathematics, it is necessary to look at the mathematical content that the teaching and learning concern first. Consequently, the first theme concerns the notions of real analysis and how this topic is acquired by students. Since the notion of justification is significant at university level

and plays a central role in the research questions, a separate section is devoted to the role of justification in mathematical analysis and how students come to acquire this particular aspect. This includes the role that proof validation plays for the acquisition of mathematical analysis and the conception of proofs held by students. After having presented the mathematical domain and having looked at theoretical considerations regarding the learning of real analysis, it is time to shift focus to the teaching of real analysis. The third theme concerns research studies dealing with characterisations of teaching practices or specific elements of teaching practices, and also research focusing on how to teach mathematical analysis in a productive way. Since the research questions focus especially on student difficulties justifying statements in real analysis, the two final themes are devoted to this aspect. Thus, the fourth theme treats how students justify mathematical statements (alone, in small groups or in collaboration with a teacher), and the difficulties they experience in proof construction situations. Since this thesis takes the perspective of justification and proving as problem solving, it is also relevant to look at what the literature has to say about problem solving in mathematical analysis, which is the fifth theme.

For some of the five themes, the research literature may appear limited. Since relevant literature exists concerning mathematical domains and educational levels other than mathematical analysis and the university level, respectively, the borderline of the literature survey is crossed in some cases.

## 2.1 Learning mathematical analysis

In a recent paper Harel [2008] addresses the question: What is the mathematics that we should teach in school? He views mathematics as composed of two sets:

The first set is a collection, or a structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the *institutionalised* ways of understanding in mathematics throughout history. The second set consists of all the ways of thinking that are characteristics of the mental acts whose products comprise the first set. [Harel, 2008, p. 490]

By institutionalised, Harel refers to ways of understanding accepted by the mathematics community. Mental acts produce products, which are called ways of understanding, but the mental acts have some characteristics, which are called ways of thinking. Learning mathematics demands both knowledge about the products of certain mental acts, but also knowledge about appropriate ways of thinking in mathematics. Harel argues that instruction often favours ways of understanding over ways of thinking, thus providing students with access to only half of what constitutes mathematics as a scientific discipline [Harel, 2008, p. 490].

### 2.1.1 The objects of real analysis

Mathematical analysis is concerned with limits, functions, continuity, differentiability, integrability, and convergence of sequences and series. The historical development of these notions extended over many centuries, and the huge amount of epistemological obstacles<sup>1</sup> related to real analysis is reflected by this [Cornu, 1991; Sierpinska, 1987]. As an example, Juter [2006a] found that university students trying to learn the concept of limit of functions go through the same stages of difficulties as can be detected in the historical development of the concept.

Contrary to calculus, the objects of real analysis are general functions, and the tasks in real analysis thus concern general functions. When university students take their first analysis course they have encountered many different specific functions and used many different algorithms and procedures related to function investigation. Studies show that university students tend to focus on procedural aspects of functions, but without knowing why the procedures and algorithms work [Eisenberg, 1991, p. 147], and that this tendency to focus on procedure is related to the students' lack of graphical understanding. They hesitate to use graphs or sketches when producing arguments, and instead they resort to analytical and algebraical forms of argumentation [Eisenberg, 1991, p. 146]. The reason could be that students in high school develop a narrow image of functions where only functions given by an explicit expression or formula can be regarded as functions and they are only used to do very simple calculations on functions (isolate unknowns and calculate different function values) [Dubinsky, 1994, p. 237-238].

The notion of limit of functions is the foundation of all the major concepts of mathematical analysis. University students find the notion of limit hard to learn, but view it as one of the most important concepts in analysis [Juter, 2005]. There are several epistemological obstacles related to the notion of limits of functions [Cornu, 1991; Sierpinska, 1987]. One of them is concerned with the question: is the limit ever attained? Studies show that some university students think that the limit cannot be reached [Juter, 2006b; Williams, 1991], resulting in the refusal of the number 0.9 being equal to 1 [Szydlik, 2000], and others think that converging sequences eventually will reach their limits [Cornu, 1991, p. 162]. Williams [1991] found that the image of a limit as something which is never reached was held by students who only found the notion of limit relevant for functions not continuous at the point of interest. Another difficulty for students is to separate between the limit value of a function,  $\lim_{x \rightarrow a} f(x)$ , and the value of the function,  $f(a)$ , [Tall, 1991; Juter, 2006b].

The epsilon- $N$  definition of limits of sequences and the epsilon-delta defini-

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<sup>1</sup> Epistemological obstacles are a sub-category of cognitive obstacles. Cognitive obstacles have to do with the difficulties students face when trying to learn new concepts. Epistemological obstacles concern those difficulties related to the *nature* of the mathematical concepts. [Cornu, 1991, p. 158] The term was originally coined by Bachelard in his work "*La formation de l'esprit scientifique*" from 1938 [Sierpinska, 1994, p. 133-134].

tion of limits and continuity of functions introduce new obstacles for students, since the definitions rely heavily on the notion of quantifiers such as ‘there exists’ and ‘for every’. These phrases have everyday meanings, which supposedly introduces difficulties for students handling them mathematically [Cornu, 1991; Monaghan, 1991; Epp, 1999]. But difficulties with quantifiers are also connected to students’ lack of knowledge about logic. Studies where university students are asked to negate statements in both everyday contexts and mathematical contexts show that students find it equally difficult to negate statements in both contexts [Barnard, 1995], and that the content of the statements influences students’ ability to handle quantifiers properly [Dubinsky & Yiparaki, 2000]. This shows that lack of knowledge of logic and proficiency in operation with logical statements also play a role in students’ difficulties with mathematics. Selden & Selden [1995] found that only 8,5% of 61 university students taking a ‘bridge’ course were able to write a correct logical translation of an informal mathematical statement.<sup>2</sup>

### 2.1.2 Mathematical understanding and concept formation

There seems to be a general agreement among researchers that an individual’s mathematical understanding of mathematical objects and concepts concerns the establishment of mental relations between mental images of mathematical concepts. Researchers use different terminologies, and details of the frameworks may vary, but overall the different notions of understanding seem to ‘move in the same direction’. The main feature is that mathematical understanding concerns building networks in the mind connecting different pieces of information. The proponents of this cognitive approach speak very little about how students or learners develop mathematical meaning. The ability to makes sense or ascribe meaning to mathematical symbols and representations must be considered as part of mathematical understanding. This aspect of understanding that relates to sense-making is in focus in theoretical perspectives dealing with the interactions between individuals and social environments where learning takes place or is initiated, e.g. in classrooms. The cognitive frameworks for understanding will be presented in this section, while some of the more socially oriented frameworks will be presented in the section about teaching practices.

Hiebert & Carpenter [1992] base their framework of understanding on the existence of external and internal representations of mathematical ideas, facts and procedures. Understanding is then believed to be a cognitive network consisting of relations between internal representations of ideas forming an inner network; “*If something is understood, it is represented in a way that connects it to a network*” [Hiebert & Carpenter, 1992, p. 75]. But the network of internal representations cannot be observed, whereas external representations and connections to some degree can be observed through experiments or observations. It

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<sup>2</sup> An example of an informal statement: For  $a < b$ , there is a  $c$  such that  $f(c) = y$  whenever  $f(a) < y$  and  $y < f(b)$ . [Selden & Selden, 1995, p. 137]

is assumed that external representations are connected in an unspecified way to the internal representations and their relations which constitutes the true image of the individual's understanding. It is also assumed that connecting external representations influences the connection between inner representations and thus affects the development of mathematical understanding. The external representations are thus both a means to examine the net of internal representations, and also a way to influence and change the internal network, giving rise to the development of understanding. The degree of understanding that a person can have of a mathematical concept depends on the number and strength of connections between various representations of that concept. An example is provided: if a person connects the written epsilon- $N$  definition of a limit of a sequence of numbers with an illustration that person's understanding of limits is believed to be richer than without this connection [Hiebert & Carpenter, 1992, p. 68].

Hanna [2000] promotes a similar representation of understanding, but is only concerned with understanding of proof and not the understanding of mathematical concepts and ideas in general. Referring to [Rav, 1999] and Yuri Manin<sup>3</sup>, she presents understanding through the metaphor of a transportation system where axioms, definitions and theorems constitute 'bus stops', and the proofs are the roads that allow a (sightseeing) bus to get from one bus stop (or important sight) to the next [Hanna, 2000, p. 7].

In the terminology used by Hiebert & Carpenter [1992] understanding is developed when new information is connected to the already existing inner network or when the network is rearranged and old connections are terminated and new ones are made. Rebuilding the network could be a result of activities such as task solving.

In the introductory chapter in the book "*Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis*" Hiebert and Lefevre talk about the relationship between skills, understanding and the development of mathematical competence : "... *skills and understandings are important because they signal two kinds of knowledge that play crucial, interactive roles in the development of mathematical competence*" [Hiebert & Lefevre, 1986, p. 23]. Skills are related to *procedural knowledge*, which contains both knowledge about the symbolic language and knowledge about procedures and algorithms for solving mathematical tasks, including non-symbolic operating strategies for solving mathematical problems. Procedural knowledge does not contain knowledge about why a proof is correct. This belongs to the category of *conceptual knowledge*. The characterisation of this kind of knowledge resembles Hiebert and Carpenter's definition of understanding [Hiebert & Carpenter, 1992] which I described before, also referred to as conceptual understanding. The aim of Hiebert and Lefevre's chapter is to state that the two different kinds of knowledge are equally important:

If we understood more about the acquisition of these kinds of knowledge and

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<sup>3</sup> From a panel discussion on 'The theory and practice of proof' at ICME-7 in 1992.



the interplay between them in mathematical performance, we surely could unlock some doors that have until now hidden significant learning problems in mathematics. (...) it now is evident that it is the *relationships* between conceptual and procedural knowledge that hold the key. The skills and understanding issue is important, to be sure, but not because instruction should choose between them. [Hiebert & Lefevre, 1986, p. 22 and 23]

Assuming, as the quote says, that instruction should not choose between teaching for understanding and practising of skill it is reasonable to discuss if students should first train procedural skills by solving a lot of exercises and later move on to problems in order to train the development of conceptual knowledge, or it should be the other way round or a mix of the two. In a cross-cultural study of primary school teachers' beliefs about effective mathematics instruction, teachers from Mainland China, USA, Australia and Hong Kong are asked about their beliefs about the relation between memorisation and understanding [Cai, 2007, p. 267]. All four groups of teachers believe that memorisation is needed in learning, but that it should come after understanding (perceived as conceptual understanding) has occurred, but as the only group of teachers, Chinese teachers believe that memorisation also can lead to understanding although they only see memorisation as a transitional stage towards understanding and not as the final goal [Cai, 2007, p. 267][Wang & Cai, 2007, p. 292-293]. This picture is also seen in the following description, although in a more uncompromising version and not substantiated through empirical evidence:

Most Chinese teachers believe in 'first memorize it, and then understand it step by step.' For example, although children do not understand why they should undertake piano finger exercises, they have to memorize them, and then understand things later. Similarly, our ability to speak our mother tongue just relies on memorization and imitation, even if we do not understand what the grammar is. In China, we usually say: 'if you want to understand something, you should practice it; even if you do not understand it well, you have to practice, too. In the process of doing you will understand things better and better.' [Zhang & Dai, 2008, p. 4]

The same discussion becomes relevant in the characterisation of understanding proposed by Skemp [1976]. Skemp proposes a characterisation of understanding as either instrumental or relational understanding. His proposal triggered a lively debate which resulted in a revised characterisation consisting of three different types of understanding [Skemp, 1979]. *Instrumental understanding* is defined as "the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works" [Skemp, 1979, p. 45]. *Relational understanding* is defined as "the ability to deduce specific rules or procedures from more general mathematical relationships" [Skemp, 1979, p. 45] whereas *formal understanding* is "the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning" [Skemp, 1979, p. 45]. In contrast to the definition of procedural and

conceptual knowledge where the two kinds are equally important it is clear from the formulation of instrumental and relational understanding that the second kind is favoured over the first kind since knowing ‘how and why’ a rule works is better than just knowing ‘how’.

Instead of viewing the different kinds of understanding as non-related parts in a static partition, Tall [1978] places emphasis on the process of developing mathematical understanding where the different kinds of understanding each play their own part. Viewing understanding as a process where all the different types of understanding play a role at different times leads the discussion away from regarding one type of understanding as ‘the best one’.

Dreyfus [1991] also talks about understanding as a process. Understanding of mathematical concepts is often the result of the learner participating in a long chain of learning activities which influences the mental processes and the creation of mental images of the concepts. Dreyfus focuses on the process of representing as one central process in the development of understanding. *“To represent a concept, then, means to generate an instance, specimen, example, image of it”* [Dreyfus, 1991, p. 31]. A representation can either be symbolic, and then it is externally written or spoken, or it can be mental and thus internal. A mental representation is a personal mental image of a concept. It is the inner ‘vision’ or image of the concept that comes to the mind of the learner when he or she is asked to think about the concept. It is possible for a person to have different competing mental representations at the same time, but this will at some point give rise to difficulties in solving situations. Learning activities such as problem solving promote the linking between mental representations of concepts, and it is of course

...desirable to have rich mental representations of concepts. A representation is rich if it contains many linked aspects of that concept. A representation is poor if it has too few elements to allow for flexibility in problem solving. [Dreyfus, 1991, p. 32]

Dreyfus explains that two professional mathematicians most likely will provide identical definitions of a concept, but that their mental representations may be very different.

Viewing acquisition of understanding as a process is a pivotal point in the model of growth in understanding of a mathematical concept proposed by Pirie & Kieren [1994]. The complete model with all its aspects is rather complex, but the authors basically argue the view point that in order to understand a mathematical concept a person must go through eight different levels. For Pirie and Kieren the process of understanding starts with activities of a procedural nature, and through such activities the person goes through more and more sophisticated levels of abstraction ending up with the ability to construct mathematical structures and ask new questions regarding the acquired concept. They picture the model as eight embedded shells which illustrates that each level contains the

previous ones and is contained in the forthcoming ones. In practice, growth in understanding is viewed by Pirie and Kieren as a non-unidirectional process where the eight levels not necessarily come in a certain order and the person learning the concept can go through one particular level several times. To go back to earlier levels in order to correct inadequate understanding is called to *fold back* [Pirie & Kieren, 1994, p. 173]. The model can be used to analyse growth in students' mathematical understanding but it is also applicable as a didactic tool. When teachers see signs of inadequate understanding at a certain level they can promote a folding back to previous levels in order for the student to acquire a more adequate understanding.

Tall and Vinner's notions of concept definition and concept image [Tall & Vinner, 1981] have strong similarities with Dreyfus's description of concept representations. Tall and Vinner use the term *concept image* to define "*the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes*", whereas the *concept definition* is composed by "*a form of words*" that describes the given concept [Tall & Vinner, 1981, p. 152]. The concept definition can be personal in the sense that the learner has constructed it for himself or it can be formal when it is the definition accepted by the mathematical community as being part of the formal theory. Tall and Vinner give an example with limits: the verbal definition of the limit of a sequence  $s_n \rightarrow s$  says that "*we can make  $s_n$  as close to  $s$  as we please, provided that we make  $n$  sufficiently large*" [Tall & Vinner, 1981, p. 153].

Besides the concept definition and concept image a person's *concept understanding scheme* also includes the usage of concepts, that is, how the person is able to use the concept in for instance a proof production situation or when generating examples [Moore, 1994, p. 252-253].

Besides identifying the notions of concept definition and concept image as important constructs in discussing the learning of mathematics and concept formation, Tall and Vinner also showed that discrepancies between a student's concept image and the associated concept definition create cognitive conflicts [Tall & Vinner, 1981]. According to their theoretical construct, a concept is acquired when a person has formed a correct concept image, and in order to possess "*deep understanding*" it is necessary (but not sufficient) for the person to be able to reproduce the concept definition [Vinner, 1991, p. 69,79]. Tall and Vinner found that students (non-mathematics majors) had difficulties learning and using a new definition of a concept (the definition of a tangent) because it caused a conflict with their concept image of the previously learned definition (the tangent of a circle) [Vinner, 1991, p. 73-78]. And when mathematically gifted high school students were asked to reconstruct a concept definition of the limit of a sequence after a summer break they used their incorrect concept images to reformulate the definition, resulting in incorrect definitions [Vinner, 1991, p. 78-79].

In a study of the development of college calculus students' concept images of limit of functions, Williams [1991] found that students possessed different (both

correct and incorrect) images at the same time. 341 students were asked to evaluate six different images of a limit of a function [Williams, 1991, p. 221]:

- **Dynamic-theoretical** A limit describes how a function moves as  $x$  moves toward a certain point;
- **Acting as a boundary** A limit is a number or point past which a function cannot go;
- **Formal** A limit is a number that the  $y$ -values of a function can be made arbitrarily close to by restricting  $x$ -values;
- **Unreachable** A limit is a number or a point the function gets close to but never reaches;
- **Acting as approximation** A limit is an approximation that can be made as accurate as you wish;
- **Dynamic-practical** A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

A majority of the students viewed the dynamical-theoretical, the unreachable, and the formal image as being true images. A majority of those students who viewed the formal image to be true also viewed the dynamical-theoretical (82%) and the unreachable image (65%) to be true. In an attempt to alter the students' concept images during four individual half hour sessions over a period of seven weeks (10 students were selected), Williams found that even though students experienced cognitive conflicts, they strongly hesitated to alter their concept images.

In her studies of undergraduate calculus students, Szydlik [2000] found that students' beliefs about calculus affected the development of correct concept images of the limit of a function. Students with an external source of conviction viewed calculus as a set of facts to be remembered, and could not appreciate or see that theory provided understanding for the use of specific procedures. Those students held incorrect and inconsistent concept images of the limit of a function (as a boundary that cannot be crossed or as unreachable). Students who viewed calculus as "*logical and consistent*" [Szydlik, 2000, p. 273] and had an internal source of conviction obtained concept images without inconsistencies. Szydlik concluded that students with an external source of conviction would not benefit from a stringent, rigorous and structured presentation, whereas students with an internal source of conviction would experience frustration with an informal form of presentation [Szydlik, 2000, p. 274].

But also the lack of computational skills can hinder the development of correct concept images. Juter [2006b] found that students' construction of proper concept images of limit of functions were obstructed by lack of abilities to manipulate algebraic expressions.

Based on an analysis in [Vinner, 1991] of the various possibilities of interplay between a person's concept definition<sup>4</sup> and the concept image during the process

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<sup>4</sup> Vinner is not always clear about when he is talking about the formal concept definition

of concept formation, Edwards & Ward [2004] studied how university students perceive the role of definitions in mathematics and how they use definitions to solve tasks in analysis and abstract algebra. They found examples of students regarding definitions as the result of a theorem (*“Once [a theorem] is proven, it becomes a definition”*, [Edwards & Ward, 2004, p. 415]). Mathematical definitions are thus regarded as delivering facts. Another student hesitated to look up the formal definitions during task solving, and she presumably perceived formal definitions as something she was supposed to be able to extract from instances. So after having seen a lot of examples or realisations of a given concept definition the formal definition was no longer found useful in a solving situation. Another finding was that if a conflict occurred between a student’s concept image and the concept definition the student would choose the concept image as a foundation for the argumentation even though the concept definition was available.

The notions of *concept* and *conception* introduced by Sfard [1991] bear strong resemblance to the notions of concept definition and concept image. In her definition a concept explains a mathematical idea presented in its formal form whereas conception is used for *“the whole cluster of internal representations and associations evoked by the concept”* [Sfard, 1991, p. 3]. In her definition, Sfard separates further between structural and operational conceptions of a concept. The two complementary ways to view a concept build on the claim that mathematical concepts can be viewed both as an object (structural conception) and as a process or processes (operational conception). In concept formation a notion is first viewed as a process, and later the student is able to see that the process can be *encapsulated* into an objects on which other processes can be applied. Sfard argues – in line with Hiebert and Lefevre – that the point is not to choose the best way to think about concepts because one way cannot function without the other [Sfard, 1991, pp. 8-10]. She argues that the operational mode of thinking can be viewed as providing the basis for understanding, since it is impossible to claim to have understood a mathematical concept if one does not possess technical skills concerning the concept [Sfard, 1991, p. 10].

The way of viewing a concept as both a process and an object, has some similarities with the notion of ‘procept’ defined by Gray & Tall [1994]. A symbol is called a procept if the symbol defines both a **process** and a **concept**. One example is the symbol of limit: *“the notation  $\lim_{x \rightarrow a} f(x)$  represents both the process of tending to a limit and the concept of the value of the limit”* [Gray & Tall, 1994, p. 120]. They hypothesise that successful mathematical thinking demands that the person can think proceptually. This means to be able to view a certain mathematical operation as both a process and as something providing a concept on which new operations can be carried out. In their paper, primary school students’ conceptualisations of numbers, counting and addition/subtraction were studied. When trying to add two numbers (e.g.  $13 + 5$ ) less able students were

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and a person’s *concept definition image*, the personal concept definition, which is something located in a person’s cognitive structure [Vinner, 1991, p. 69].

inclined to count while more able students could operate with numbers as a concept and use known facts or derive and use new facts (e.g. treating 13 as a number and counting 5). So the procedures used by less able students get very quickly very complicated and time demanding, so the conclusion is that less successful students are actually trying to carry out a more complicated line of thought than more successful students.

Vinner [1997] proposes a framework for analysing mathematical behaviour. In his view, the educational goal is to make students perform mental processes of a certain kind which he calls conceptual thinking (thinking required for obtaining conceptual knowledge (Hiebert) and relational understanding (Skemp)) and analytical thinking (the desired form of thinking during problem solving). Conceptual and analytical thinking result in conceptual and analytical behaviour which can be observed. Vinner hypothesises that students are capable of behaving in a way that at first sight looks like conceptual or analytical behaviour, but which is actually not the product of conceptual and analytical thinking. He uses the terms pseudo-conceptual and pseudo-analytical behaviour to describe these types of behaviours. Pseudo-behaviours are results of the social environment that the student is a part of in a given teaching-learning situation. Vinner speculates that pseudo-behaviours are caused mainly by a student's eagerness to give the right answer to a given question using a minimal amount of effort. Pseudo-behaviours are typically faster, because they are a result of "*spontaneous, natural, but uncontrolled associations*" [Vinner, 1997, p. 125], and often they actually lead the student to the right answer to a question, but without giving rise to the kind of mental processes intended.

Lithner [2000b] examines forms of analytical behaviours when undergraduate students solve calculus tasks. Instead of analytical and pseudo-analytical behaviour he uses the terminology of plausible reasoning and reasoning based on established experiences. Plausible reasoning expresses a degree of certainty in the reasoning compared to analytical behaviour, and reasoning based on established experiences is a subset of pseudo-analytical behaviour. Lithner found that university students very often used reasoning based on established experiences, but also that the students had little success in using this type of reasoning. The difficulties that the students experienced during the task solving could be ascribed to the refusal to use plausible reasoning.

### 2.1.3 Mathematical competencies

"*To master mathematics means to possess mathematical competence*" [Niss, 2003, p. 119]. But what does mathematical competence mean? In the quote by Hiebert & Lefevre [1986] mentioned above, they state that both understanding and skills are required in order for a person to develop mathematical competence (see page 21). This does not explain what mathematical competence is, only what is necessary in order to obtain it.

In a report<sup>5</sup> commissioned by the Danish Ministry of Education, Niss et al. [2002] defines what it means to possess mathematical competence: “*mathematical competence means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role*” [Niss, 2003, p. 120]. And in order to possess mathematical competence, Niss and co-workers agree with Hiebert and Lefevre that factual knowledge and technical skills are necessary (but not sufficient).

Niss et al. [2002] develop the notion of mathematical competency in order to provide a more operational definition of what it means to master mathematics: “*a mathematical competency is a clearly recognisable and distinct, major constituent of mathematical competence*” [Niss, 2003, p. 120]. Niss et al. [2002] identify eight different mathematical competencies encompassing all mathematical competence and clarify in details how these competencies unfold at various educational levels and within different educational subjects where mathematics plays a role (including also for instance in the education of electricians). The eight identified competencies are divided in two main groups, to “*ask and answer in and with mathematics*” and to “*deal with and manage mathematical language and tools*” [Niss et al., 2002, p. 44][Niss, 2003, p. 120-121]. The division is illustrated by the ‘competence flower’ in figure 2.1.

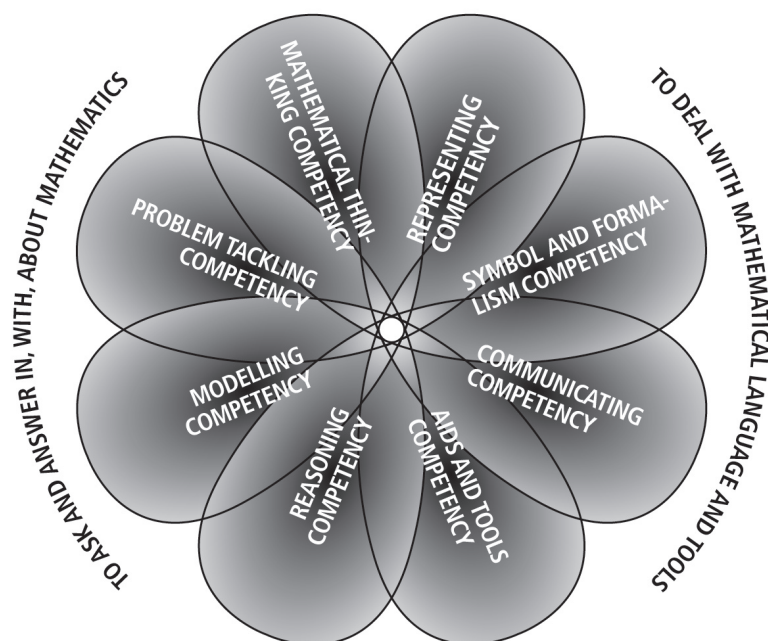
Each competency is composed of two parts, an analytical part (understanding, interpreting, examining, and assessing mathematical phenomena and processes) and a productive part (active construction and carrying out of mathematical processes), and these two aspects of each competency can be found in the following description of the eight competencies [Niss, 2003, p. 120-121]:

**Thinking mathematically** (mastering mathematical modes of thought)  
such as

- *posing questions* that are characteristic of mathematics, and *knowing the kinds* of answers (not necessarily the answers themselves or how to obtain them) that mathematics may offer;
- understanding and handling the *scope* and *limitations* of a given *concept*;
- *extending* the scope of a *concept* by *abstracting* some of its properties: *generalising results* to a larger class of objects;
- *distinguishing* between different *kinds of mathematical statements* (including conditioned assertions (‘if-then’), quantifier laden statements, assumptions, definitions, theorems, conjectures, cases).

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<sup>5</sup> The aim of the report was to “*explore the terrain of mathematics teaching and learning*” [Niss, 2003, p. 118] in order to deal with identified problems and challenges within the Danish mathematics education system, e.g. challenges such as the decrease in number of students entering educational programs with a high level of mathematics, transition difficulties between different levels in the educational system and problems with assessment.



**Figure 2.1** Illustration of how the eight different mathematical competencies are related. The flower symbolises that one competency depends upon the seven other competencies but that it is impossible to reduce it to one of the others. It is not possible to possess only one competency because the other competencies are necessary in order to acquire, possess and carry out a given competency. Mathematical competence can be described exhaustively through the union of the eight competencies. How the eight competencies are divided into two main groups can also be seen in the illustration. (The illustration is an unpublished English version of the Danish illustration found in [Niss et al., 2002, p. 45].)

### Posing and solving mathematical problems

such as

- *identifying, posing, and specifying* different kinds of mathematical *problems* – pure and applied; open-ended or closed;
- *solving* different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or by oneself, and, if appropriate, in different ways.

### Modelling mathematically (i.e. analysing and building models)

such as

- *analysing* foundations and properties of *existing models*, including assessing their range and validity;
- *decoding* existing models, i.e. translating and interpreting model elements in terms of the 'reality' modelled;
- *performing active modelling* in a given context
  - structuring the field
  - mathematising



- working with(in) the model, including solving the problems it gives rise to
- validating the model, internally and externally
- analysing and criticising the model, in itself and vis-à-vis possible alternatives
- communicating about the model and its results
- monitoring and controlling the entire modelling process.

### Reasoning mathematically

such as

- *following* and *assessing chains of arguments*, put forward by others;
- *knowing* what a mathematical *proof is* (not), and how it differs from other kinds of mathematical reasoning, e.g. heuristics;
- *uncovering* the *basic ideas* in a given line of argument (especially a proof), including distinguishing main lines from details, ideas from technicalities;
- *devising* formal and informal mathematical *arguments*, and *transforming* heuristic arguments to valid proofs, i.e. *proving statements*.

### Representing mathematical entities (objects and situations)

such as

- *understanding* and *utilising* (decoding, interpreting, distinguishing between) different sorts of representations of mathematical objects, phenomena and situations;
- understanding and utilising the *relations between different representations* of the same entity, including knowing about their relative strengths and limitations;
- *choosing* and *switching* between representations.

### Handling mathematical symbols and formalism

such as

- *decoding* and *interpreting symbolic and formal mathematical language*, and understanding *its relation to natural language*;
- understanding the *nature* and *rules of formal mathematical systems* (both syntax and semantics);
- *translating* from *natural language* to *formal/symbolic language*;
- *handling* and manipulating statements and *expressions* containing *symbols* and *formulae*.

### Communicating in, with and about mathematics

such as

- *understanding other's* written, visual, or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content;
- *expressing oneself*, at different levels of theoretical and technical precision, in oral, visual, or written form, about such matters.

### Making use of aids and tools (IT included)

such as

- *knowing* the *existence* and *properties* of various tools and aids for mathematical activity, and their range and limitations;
- being able to *reflectively use* such aids and tools.

It should be stated that the competence perspective is behavioural, and does not intend to address or describe any mental arrangements taking place within individual persons.

Since a person has to have relevant mathematical knowledge and computational skills in order to possess mathematical competence it follows that a person needs to have knowledge about, for instance, terminology and specific computational methods related to a specific area in order to be able to possess any of the competencies in relation to that area. For instance, in differential calculus it is necessary for the learner to know the terminology associated with this area (knowing that the symbol  $\frac{dx}{dt}$  stands for the differential quotient) and to possess the necessary computational skills (being able to differentiate  $f(t) = t^2$ ).

The development of mathematical competence can only take place in dialogue with a mathematical subject matter, and in relation to actual and potential mathematical challenges [Niss, 2003, p. 123]. But can all the eight competencies be practised equally well within any mathematical subject matter? In principle the answer is yes, but some domains are more suited for practising a specific competency than others. For instance, functions in general are very well suited for developing the modelling competency whereas the symbols- and formalism competency would be well taken care of through working with abstract algebra. Since the choice of subject matter does not follow directly from the competencies, the relationship between the competencies and different mathematical topic areas can be represented by a matrix with different mathematical topics (numbers, arithmetic, geometry, functions, ...) as one dimension and the competencies as the other [Niss et al., 2002, p. 114][Niss, 2003, p. 122]. A third dimension could have been the educational level but this is not explicitly suggested in [Niss et al., 2002] nor in [Niss, 2003], but has been proposed in conference talks.

Regarding the assessment of an individual's possession of a specific competency three aspects or dimensions are considered [Niss, 2003, p. 123]: *degree of coverage* describes to which degree an individual masters the different characteristic aspects of a certain competency; *radius of action* concerns the spectrum of contexts and situations where the competency can be activated by the individual, and *technical level* describes how conceptually and technically advanced the entities and tools are with which the individual can activate the competency. Thinking of the three aspects as a "*competency volume*", the metaphor implies that if any of the three dimensions are zero the volume is zero, and two equal sized volumes can be obtained with different values of the three dimensions.

As mentioned in [Niss, 2003, p. 122] the above description of mathematical competence is not exclusive. Another set of components might provide a satisfactory description if not with the same goal then to fulfil some other goals.

This is exactly what The National Research Council (NRC) intended with the report *“Adding it up: Helping children learn mathematics”*, where the notion of *mathematical proficiency* is defined in order to describe the goals of mathematical learning at primary level [Kilpatrick et al., 2001, p. 116]. Mathematical proficiency is composed of five competencies or strands [Kilpatrick et al., 2001, p. 116]:

- **Conceptual understanding** – comprehension of mathematical concepts, operations, and relations.
- **Procedural fluency** – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
- **Strategic competence** – ability to formulate, represent, and solve mathematical problems.
- **Adaptive reasoning** – capacity for logical thought, reflection, explanations, and justification.
- **Productive disposition** – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

There are of course several differences between the two formulations, but two conspicuous differences are that the NRC-formulation contains a strand concerning students beliefs, perception and view of mathematics and mathematical enterprise (the fifth strand), and omit the ability to handle mathematical tools, including IT.

#### 2.1.4 Summary

There are lots of dichotomies in the research literature concerning acquisition of mathematical knowledge, mathematical understanding and the formation of mathematical concepts. Greatly simplified, the literature agrees upon an understanding of mathematical knowledge as composed of two poles, one related to mathematical concepts and objects, and the other to mathematical processes and procedures.

An individual’s mathematical understanding is believed to be related to the development of an internal cognitive network where the nodes are concepts, ideas, facts, procedures, etc.. The more connections the network has the deeper mathematical understanding the person possesses.

In mathematics, the concepts and objects are given through clearly stated definitions. The cognitive representations of the concepts are not just an image of these concept definitions. The person also makes images of the concepts that go beyond and are at odds with the mathematical definition. Discrepancies between a person’s concept image and the formal concept definition cause learning difficulties.

The notion of mathematical competence or mathematical proficiency is a way to describe what it means for a person to be ‘good at math’. These descriptions

can serve as analytical tools for descriptive purposes or for normative purposes in relation to educational planning.

## 2.2 Acquisition of the role of the justification of mathematical statements in mathematical analysis

In this section I present the literature concerning students' acquisition of the notion of proof in mathematics, which includes how students obtain acceptance of already completed proofs. In section 2.4 the literature concerning students' independent attempts and efforts to construct proof is treated.

### 2.2.1 Argumentation, justification, and proof

Without doubt, all mathematicians would agree that mathematical proof is one of the cornerstones of mathematical enterprise [Hanna & Jahnke, 1996; Harel & Sowder, 2007], and as Hemmi writes: “*proof constitutes the means for justifying knowledge in mathematics*” [Hemmi, 2006, p. 16]. Different philosophical schools hold different epistemological and ontological views on proof and on what constitutes a valid proof [Hemmi, 2006, p. 16-21], and different contexts use different descriptions of proof: a proof is “*an argument that convinces qualified judges*”, while proof in the definition founded on logic is “*a sequence of transformations of formal sentences, carried out according to the rules of predicate calculus*” [Hersh, 1993, p. 391].

There is a difference between a proof and the presentation of the proof, when it comes to the degree of rigour. The amount of rigour necessary in a presentation of a proof is a question of context and is thus socially contingent [Mamona-Downs & Downs, 2005, p. 387][Bell, 1976, p. 24-25]. Among professional mathematicians studies show that rigour is not primary when it comes to the acceptance by the mathematics community of a proof [Hanna & Jahnke, 1996, p. 878-879]. Other factors such as the importance of the theorem, the degree of understanding that the proof contains, and that the arguments are convincing and of a familiar type carry more weight than rigour [Hanna & Jahnke, 1996, p. 879]. And in mathematics courses, the teacher's acceptance of a proof depends on the textbook context in which the proof task is placed, and for didactical reasons the teacher might even reject one student's proof if he or she believes that the student is capable of providing a better proof, for instance a more rigorous one.

When studying the argumentation structures students use, some researchers have found it useful to apply the model of argumentation proposed by Toulmin [1969], for instance [Hoyles & Küchemann, 2002; Pedemonte, 2002; Stephan & Rasmussen, 2002; Knipping, 2003; Alcock & Weber, 2005]. In a reduced version of the model, Toulmin talks about claims, warrants, and data. The starting point for any (sensible) argumentation process must be the proclamation of a *claim*. It would be natural for the person promoting the claim to have some kind of

evidence or *data* justifying the claim. The data need not be enough to persuade another person about the truth of the claim so there is a need to have some kind of inference rules or *warrants* connecting data and claim. [Toulmin, 1969, p. 97-100]

Although this conceptualisation of an argumentation process seems to be enough to describe mathematical thinking, Inglis et al. [2007] and Jahnke [2008] argue that this reduced version is not sufficient to capture the argumentation processes observed in problem solving situations. An extended version is needed, which takes into account the notion of *backing* up the warrants, a modal *qualifier* expressing the warrants' applicability to the particular situation/claim, and a *rebuttal* stating conditions under which the claim might not hold [Toulmin, 1969, p. 101-107]. In an ideal mathematical argumentation situation there is no need to provide backing since the truth of the warrants has already been accounted for, for instance through logic. The modal qualifier is also redundant since the warrants either are applicable or not, and furthermore a mathematical claim cannot be proven if exceptions exist, and this removes the need for a rebuttal.

To 'back-up' the claim that the extended version is needed, Jahnke [2008] invokes the notion of open and closed general statements. In every day thinking we operate with open general statements. These are statements that people perceive as true in general even though exceptions might occur.<sup>6</sup> In mathematics, only general closed statements are considered. These do not allow exceptions. Jahnke [2008] argues that students mistakenly assume that the rules of every day thinking also apply in mathematics, and in his view this explains why some students find it difficult to understand that only one exception dismisses a mathematical claim.

When a person states a claim he or she can either be certain of the truth of the claim and then the claim stated is considered as a *fact* or he or she can be uncertain of its truth and then the claim is a *conjecture*. The act of proving is the process where the person removes his or her (or other people's) doubts about the truth of the claim, that is the process of transforming the conjecture into a fact. [Harel & Sowder, 2007, p. 808]

In her studies, Pedemonte studies the relationship between argumentation and proof [Pedemonte, 2002, 2007, 2008]. Argumentation is the process leading to the formulation of a conjecture while the proof establishes the truth of the conjecture. She found that, when students solve open problems in algebra (where they are supposed to construct a proof) processes of argumentation can make the process of proving easier for students [Pedemonte, 2008].

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<sup>6</sup> Jahnke gives an example: a girl is convinced that her father will come home every single evening at 6pm since his office closes at 5pm and it takes an hour by train. But she does not expect it to happen every evening (e.g. she knows the train might be late) [Jahnke, 2008, p. 364].

### 2.2.2 Why teach proof?

There are a lot of different functions attached to the notion of proof. Bell [1976] attaches three different functions to the notion of proof of a mathematical statement: *Verification/justification* concerns the truth of the statement, *illumination* reveals why the statement is true<sup>7</sup> (if it is a good proof), and last *systematisation* where mathematical results are organised into a deductive system of major concepts, axioms, theorems and derived corollaries [Bell, 1976, p. 24]. Five additional functions can be added to the list: *discovery* of new results, *communication* of mathematical knowledge, *exploration* of the meaning of a definition or the consequences of an assumption, and *incorporation* of a well-known fact into a new framework [de Villiers, 1990, p. 18][Hanna, 2000, p. 8]. In mathematics research the most important function of proof might be to convince colleagues that a proposed conjecture is correct, while in a mathematics classroom the students are already convinced about the truth of proposed theorems, as long as the theorems are presented in a textbook. Instead, they need the proofs to be explanatory in order to understand why a theorem is true [Hersh, 1993, p. 396].

This view is shared by Hanna [2000]. From an educational point of view Hanna regards explanation as the most significant function of proof since “... *the key role of proof is the promotion of mathematical understanding*” [Hanna, 2000, p. 5-6, my footnote].

Almeida [2000] agrees with Hanna that proofs convey understanding and states that “*Mathematical proof provides a warrant... for mathematical knowledge and is an essential activity in doing and understanding mathematics*” [Almeida, 2000, p. 869]. But it is not unimportant what a person does to try to understand a proof. The group of Bourbaki questioned that the form of activity where the student persuades herself about the accuracy of each step in the proof leads to understanding:

Indeed every mathematician knows that a proof has not really been ‘understood’ if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one. [Bourbaki, 1950, p. 223, footnote]

Alcock & Weber [2005] argue that proof validation consists of more than confirming that all the steps in the proof are true. It is necessary to check the warrants<sup>9</sup> for making each argument.

<sup>7</sup> A proof can be *valid* even though it does not have a high degree of illumination.

<sup>8</sup> As described on page 20, Hanna views understanding through the transportation metaphor as a network of roads in a transportation system where definitions and theorems are bus stops and the proofs constitute roads connecting various definitions and theorems [Hanna, 2000, p. 7].

<sup>9</sup> Warrants as defined by Toulmin [1969], see page 33.

Selden & Selden [1995] provide an elaboration of what the activity of validating proof means and accomplishes. The process of proof validation consists of activities where the student examines and convinces herself about the truth of the proof. This implies both an investigation about the correctness of the proofs itself and if the proof actually proves the given theorem (in order for the student to convince herself about the latter she has to make, what the authors call, a *proof framework*). The activities can include reading the proof, asking questions about the proof, making subproofs for claims such as ‘it is trivial to see that...’ [Selden & Selden, 1995, p. 127]. According to Selden and Selden validating proofs strengthens the students’ *statement image* of a theorem [Selden & Selden, 1995, p. 133]. This notion is a direct elaboration of Tall and Vinner’s notion of a concept image, but instead of talking about concepts such as functions, continuity and limits, Selden and Selden focus on students’ (and mathematicians’) mental images of statements such as definitions, theorems and conjectures. A statement image is thus the mental network associated with a given statement. Statement images

...are meant to include all of the alternative statements, examples, nonexamples, visualizations, properties, concepts, consequences, etc., that are associated with a statement. Such associations can arise from noticing relationships, such as seeing an example which illustrates a theorem, from repetition, such as using a theorem many times on one type of problem; or from affect, such as discovering a proof technique after many attempts. [Selden & Selden, 1995, p. 133]

By validating a proof, the student reinforces the statement image of the particular theorem by establishing or reinforcing connections between various representations of the statement. But theorems can also be seen “*as carriers of the complex relationships between concepts*” [Selden & Selden, 1995, p. 134], so proof validation can enhance conceptual understanding.

Inspired by a paper by Rav ‘Why do we prove theorems?’ [Rav, 1999], Hanna & Barbeau [2008] study how proofs act as “*bearers of mathematical knowledge*”. They argue that proof demonstrations in class might be used to introduce students to important mathematical strategies and methods useful in problem solving (e.g. completing the square in quadratic polynomials). This view extends the role of proof in education beyond justification and explanation/understanding.

### 2.2.3 Students’ conceptions of proof

Studies show that many university students hold very different and often wrong or insufficient conceptions of proof and thus of what it means to prove mathematical statements [Selden & Selden, 1995; Harel & Sowder, 1998; Jones, 2000; Almeida, 2000; Knuth, 2002]. An often cited framework for discussing students’ proof conceptions is the framework by Harel and Sowder based on the notion of *proof schemes* [Harel & Sowder, 1998, 2007]:

A person's (or a community's) proof scheme consists of what constitutes ascertaining<sup>10</sup> and persuading<sup>11</sup> for that person (or community). [Harel & Sowder, 2007, p. 809] (My footnotes)

In the literature these two aspects of proof schemes (ascertaining and persuading) are also referred to as the *private/personal* and *public* parts of proof production [Segal, 2000; Raman, 2003; Mejia-Ramos & Tall, 2005].

The notion of proof and proof schemes are specific examples of the more general notions of ways of understanding and ways of thinking proposed by Harel [2008] (the notions were briefly mentioned on page 18), and perceiving the proving process as a mental act

... we have here a triad of concepts: proving act, proof, and proof schemes. A proof is a cognitive product of the proving act, and proof scheme is a cognitive characteristic of that act. Such a characteristic is a common property among one's proofs. ... [Harel, 2008, p. 489]

The development of the framework of proof schemes was based on data from teaching experiments in geometry and linear algebra, but are certainly also applicable for topics such as real analysis. The framework is both a description of students' conceptions of mathematical justification and proof, and a characterisation of their reasoning in justification processes. The framework divides students' proof schemes into three main categories each with several sub-categories: external conviction schemes, empirical schemes and deductive<sup>12</sup> schemes.

The first category concerns those conceptions and processes where students need external sources to provide conviction of the truth of a statement. This could for instance be the professor or the textbook (authoritarian scheme). A well-known sign of an authoritarian scheme is for instance when students avoid asking the professor to explain his way of thinking during the process of justifying statements in class. Often students accept proofs just because they contain symbolic manipulations, or they think that proofs need to be based on symbolic manipulations (symbolic scheme). Finally, students sometimes ascribe too much importance to the ritual element of proof, for instance they accept (wrong) justifications because the justifications follow a ritual way of presenting proof (ritual scheme).

In the second category, the empirical proof schemes, students accept or construct proofs based on specific instances (inductive schemes). They confuse proof by contradiction with 'proof by example', which unless it is proof by induction

<sup>10</sup> "Ascertaining is the process an individual (or a community) employs to remove her or his (or its) own doubts about the truth of an assertion" [Harel & Sowder, 2007, p. 808].

<sup>11</sup> "Persuading is the process an individual or a community employs to remove other's doubts about the truth of an assertion" [Harel & Sowder, 2007, p. 808].

<sup>12</sup> Since the authors in their recent presentation of the framework, [Harel & Sowder, 2007], have termed the third category *deductive schemes* I use this term instead of *analytical schemes* which was used originally in [Harel & Sowder, 1998].



(or if there are only a finite number of elements to examine) is not acceptable as a proof. Geometry tasks often contain a drawing of some geometrical object that the student is asked to prove statements about. If the student bases his or her argumentation on the actual appearance of the objects in the task, and is not capable of anticipating results of object transformation the student has a perceptual scheme of proof.

The third category, the deductive proof schemes, is divided into transformational and (modern) axiomatic proof schemes. In contrast with the case of the perceptual scheme, a student having a transformational scheme is capable of making transformational actions on mental images of mathematical objects. The students' actions are goal oriented and the results of the transformations can be anticipated, and are used in deductions. Within this scheme lie two sub-categories, the causal and the Greek axiomatic schemes. A student who is aware of the meaning of axioms and the fundamental role they play in mathematics has an (modern) axiomatic scheme. This sub-scheme has the same characteristics as transformational schemes but in addition the students should realise that the collection of mathematical results (definitions, theorems, etc.) are determined by the collection of axioms and he or she should be capable of investigating the implications of varying the set of axioms.

Harel and Sowder state that university students as a goal should show some sign of having developed deductive proof schemes [Harel & Sowder, 1998, p. 277]. Within the deductive scheme, the sub-categories are believed to be dependent, such that the acquisition of an axiomatising scheme requires the student to possess a structural proof scheme. Even though the external conviction schemes and the empirical schemes have some independent value, the question is of course whether or not the first two schemes are necessary prerequisites for the acquisition of deductive proof schemes. The authors hope that this is not the case, since most university instruction seems to start at a higher level where only the deductive schemes is promoted [Harel & Sowder, 1998, p. 277].

Studies confirm that university students in mathematics often develop external conviction and empirical proof schemes [Knuth, 2002; Housman & Porter, 2003], but also that students' proof schemes can be enhanced through carefully planned instruction [Sowder & Harel, 2003].

An aspect of proof that the framework of proof schemes does not emphasise is that students do not necessarily think that arguments that convince themselves also can be used to convince others. In her study, Segal [2000] found that students consider empirical arguments to be convincing privately, but do not find them publicly convincing (ascribe validity to them). When it comes to deductive arguments, there does not seem to be a distinction between private and public value, either the students viewed the deductive argument as both personally and publicly acceptable or they did not.

The difference between privately and publicly convincing arguments, and how students and professors view the connection between the two different aspects of

proof are studied by Raman [Raman, 2002, 2003]. Using task based interviews<sup>13</sup>, she found that professors use privately convincing arguments to produce publicly acceptable argumentation (rigorous proof), while university students fail to make this connection. For them, making publicly acceptable proofs are the same as “*creating something out of nowhere*” [Raman, 2003, p. 321]. If they do not have any explicit definitions or theorems that they can combine, they get stuck, even though they might have a privately held conviction that the statement is true.

### 2.2.4 Summary

To understand the role of justification and the role of proof in mathematics is very important, since it is a fundamental characteristic of the mathematical enterprise and one of the things that separate mathematics from any other subject.

Through proof mathematical knowledge is justified and this is how professional mathematicians use proof. But in relation to mathematical instruction, proof should also convey understanding of the mathematical concepts and ideas. This is apparently not an easy task and many students have difficulties in understanding the role of proof and in determining what constitutes valid proofs. Instruction at university level should aim at developing deductive proof schemes, but often students end up having only acquired empirical proof schemes.

## 2.3 Teaching mathematical analysis

This section has a dual aim. One aim is to make a presentation of theoretical frameworks applicable for characterising teaching in mathematical analysis and the other is to present some of the literature concerning (effective) ways of teaching mathematical analysis.

Niss [1996] provides an analysis of the justifications and goals of mathematics teaching during the 20th century. There are three categories of arguments justifying the presence of mathematics education in a society. Mathematics education promotes

the technological and socio-economic development of society; the political, ideological and cultural maintenance and development of society; the provision of individuals with prerequisites which may assist them in coping with private and social life, whether in education, occupation, or as citizens.  
[Niss, 1996, p. 22]

It is not at all clear how mathematics teaching should be conducted in order to fulfil the goals lying inherent in these arguments. Which mathematical topics should be selected and how should they be taught in order to make students able to participate in and contribute to the development of society?

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<sup>13</sup> The task presented to the interviewees is: prove that the derivative of an even function is odd.

Consulting the research literature in undergraduate mathematics education, the scientific need for addressing the justification problem related to teaching mathematical analysis seems non-existing. It has not been possible to find a single paper discussing or even just mentioning the relevance of teaching mathematical analysis at the undergraduate level. Instead, the focus is placed on the way mathematical analysis is being taught (descriptive) and how it ought to be taught (normative).

According to Schoenfeld [1982] mathematics teaching should strive at teaching students to think – defined as the thinking needed to solve unfamiliar mathematical problems. Romberg [1994] sees mathematical problem solving as the core of doing mathematics and makes the analogy, that if mathematics students only acquire “*knowledge about*” mathematics and never get to “*do*” mathematics (which he defines to be the same as solving problems) it is the same as if a violinist was supposed to learn to play the violin only by listening to other violinists and learning the theory of music [Romberg, 1994, p. 289-290].

Being able to do mathematics is by others viewed as more than being able to solve mathematics problems. The notion of mathematical competence proposed by Niss et al. [2002] (see page 27) is a way to describe what it means to master mathematics, and here solving mathematical problems is only one of eight competencies that a student should possess and the development of which mathematics teaching thus should aim at.

### 2.3.1 Characterising mathematics teaching practices

Teachers’ views and beliefs about mathematics influence their way of teaching, which has been documented in the research literature [Thompson, 1992; Aguirre & Speer, 2000]. The Teacher Model Group at Berkeley, initiated by Schoenfeld, works on modelling teaching practices with the “*immodest*” long-term goal to be able to describe, explain and predict teachers’ actions in any mathematics classroom [Schoenfeld, 2000, p. 244, 249]. The model can be used to make a fine-grained analysis of a particular teaching lesson, taking the teacher’s beliefs and goals for the teaching into account. The model operates with the following terms [Schoenfeld, 2000, p. 250-253]:

A teacher wants to achieve certain goals through his or her teaching. There are goals on many time-scales, e.g. overarching goals for the students development over the course of weeks, month, years, or local goals for a short-term interaction with students. Goals can have different characteristics, such as being epistemologically oriented, content-oriented, or socially oriented.

When a teacher prepares a lesson, he or she makes a (sometimes written) lesson plan that contains the structure of the lesson (e.g. first recapitulate last lesson, introduce new concepts, do some exercises, and end with student presentations). The teacher might have a more detailed unarticulated plan, a lesson image, of what kinds of events could take place in the lesson and how he or she might react to them. If a teaching goal is part of a lesson image it is pre-determined,

but it can also emerge during instruction caused by unforeseen student reactions. To a specific goal there is a set of actions, an action plan, that the teacher intends to carry out in order to accomplish the goal.

But one thing is articulated/unarticulated plans and goals, another is what happens in the particular teaching situation. Action sequences describe what takes place in the classroom, and often there is an accordance between the action sequences and the teacher's unarticulated lesson image [Schoenfeld, 2000, p. 251].

In the two studies [Zimmerlin & Nelson, 2000] and [Schoenfeld et al., 2000], the Teacher Model Group studies the usefulness of the model/framework. The model was able to describe and explain two very different teachers' actions and choices made during two very different lessons. In [Zimmerlin & Nelson, 2000] the focus was on the importance of the teacher's lesson image when analysing the lesson. They found that knowing the lesson image of the teacher made it possible to explain the teacher's actions in the classroom, and they found an accordance between his goals and actions. But when the action sequences deviated from his lesson image his action plans were inadequate to handle the situations appropriately according to his goals.<sup>14</sup>

In [Schoenfeld et al., 2000] the situation was different since the teacher was very experienced and the topic was non-traditional. Here the description and explanation of the teacher's goal driven actions were in focus. Through applying the model it was possible to explain the teacher's action on the basis of his goals – both pre-determined and emergent goals, the last due to students' unexpected actions in class.

In relation to teaching practices with a high focus on proof, Hemmi [2006] found different views on how and why to include proof demonstrations in class. From interviews with university professors, Hemmi identifies three different teaching styles based on professors' beliefs about teaching mathematics and mathematical proof; the progressive style (*"I don't want to foist the proofs on them"*), the deductive style (*"It is high time for students to see real mathematics"*), and the classical style (*"I can't help giving some nice proofs"*) [Hemmi, 2006, p. 82]. Since the three styles have been constructed based on interviews, it is not possible – based on this study – to conclude how teachers' belief actually influence their teaching practice.

Cobb and co-workers propose a theoretical perspective allowing for an analysis of individual students' learning processes in social settings, the mathematics classroom [Cobb et al., 1997]. The framework has been developed during the authors' implementations and subsequent studies of the consequences and results of teaching experiments done in first grade [Cobb et al., 2001] and seventh grade classrooms [Cobb, 2000b]. The theoretical considerations behind the framework draw on both constructivism and sociocultural theory. Coming from the constructivistic tradition where learning is viewed as cognitive processes involving

<sup>14</sup> The teacher wanted the students to discover by themselves that  $x^0 = 1$ , but ended up telling them the identity.

only the individual student's mental activities, Cobb wanted – for both theoretical and pragmatical reasons – to expand the perspective in order to be able to account for students' mathematical development as they participate in communal practices, particular in mathematics classrooms. The theoretical reasons being obvious, the pragmatical reasons were caused by a wish to help teachers revise their instructional practices and the framework developed emerged as a tool in that process. [Cobb et al., 1997]

Cobb and co-workers do not regard the individual cognitive perspective as superior to the social perspective or vice versa. In the *emergent perspective*<sup>15</sup> the connection between the individual and the social is indirect so that the social environment does not determine the individual development but supplies the means and constraints [Cobb et al., 1997, p. 152]. Supplying the means and constraints are not the same as determining the development, and that makes sense given that mathematical development of two different students could not possibly be expected to be the same although they participate in the same teaching practice.

The underlying assumption in the framework is that analysing social aspects – the elements listed in the left column of the matrix in figure 2.2 – provides information about individual psychological constructions – the elements in the right column of the matrix.

**Social norms** describe those characteristics and regularities in the classroom activities that are not specifically connected to mathematics. A social norm could be that only the teacher and not the students was 'allowed' to ask questions. This norm would not be restrictive to mathematics but could occur in any classroom regardless of the subject matter to be taught. So social norms have to do with the activity of participants in any communal practice. Other examples of social norms could be that students are expected to explain and justify claims, to make indications of agreement or disagreement and to question alternatives when interpretations are conflicting [Cobb, 2001, p. 464-465].

**Sociomathematical norms** have to do with norms related to the fact that the subject matter is mathematics. This could for instance be the establishment of norms about what counts as a different, sophisticated or efficient mathematical solution and an acceptable mathematical explanation. These norms were found in the study of a first grade classroom [Cobb et al., 2001, p. 124]. Cobb and co-workers conjecture that the mathematical beliefs and values of the students are the psychological correlates of sociomathematical norms. So *“that in guiding the establishment of particular sociomathematical norms, teachers are simultaneously supporting their students' reorganisation of the beliefs and values that constitute*

<sup>15</sup> Cobb [1994] uses the term *emergent perspective* about a perspective that regards mathematical development as individual construction but at the same time takes into account that the individual most of the time is placed in social environments. Adherents to this perspective argue *“that neither an individual student's mathematical activity nor the classroom microculture can be adequately accounted for without considering the other”* [Cobb, 1994, p. 15].

Social Perspective	Psychological perspective
Classroom social norms	Beliefs about own role, others' role, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical interpretations and activity

**Figure 2.2** Interpretive framework for analysing classroom microculture, from [Cobb et al., 1997, p. 154]. The framework is founded on the assumption that social and psychological aspects interact and are dependent. For instance, the development of sociomathematical norms affect students' mathematical beliefs and values, and vice versa. The framework has been presented in several papers with small adjustments in the formulation of the label for the psychological counterpart to the classroom mathematical practices. One finds the following formulations “*Mathematical interpretations and activity*” [Cobb et al., 1997, p. 154], “*Mathematical interpretations and reasoning*” [Cobb et al., 2001, p. 119] or “*Mathematical conceptions*” [Cobb, 2000a, p. 159].

*what might be called their mathematical dispositions*” [Cobb, 2000b, p. 71].

**Mathematical practices** are concerned with the emergence of mathematical content [Cobb, 2001, p. 465]. This could be the establishment of certain ways to perform mathematical operations – for instance ways to count – or the ways in which students reason mathematically when they try to solve tasks. Shifts in the mathematical practices of the students indicate that a probable shift in the students' perception of the involved mathematical concepts have occurred as well.

As described above, the right column in the matrix contains the psychological consequences of the establishment and negotiation of the norms and practices in the ‘social’ column. The claim that the different aspects in the two perspectives are related in this particular way is a conjecture put forward by Cobb and co-workers and are as they write “*open to empirical investigation*” [Cobb et al., 2001, p. 124].

Although data from observations in a primary classroom was used in the development of the framework, the framework is general and can be used to analyse the establishment of norms and mathematical practices at higher educational levels. This was for instance done in [Yackel et al., 2000] and [Stephan & Rasmussen, 2002], where the social and sociomathematical norms, and the establishment of mathematical practices, respectively, were studied in undergraduate university classrooms dealing with differential equations. In [Yackel et al., 2000]

a traditional course on differential equations was compared to an experimental course, where the professor deliberately tried to establish a certain kind of social and sociomathematical norms. The experiment succeeded in establishing social norms, where students were expected to explain their thinking and also to try to make sense of the thinking of other classmates. Explanations had to be grounded in some interpretation of the rates of change, which was seen as a sociomathematical norm. Using data from the same teaching experiment, Stephan & Rasmussen [2002] studied the establishment of mathematical practices. In order to document when a mathematical idea was established as a mathematical practice, they used Toulmin's argumentation model (see page 33). When the students no longer provided warrants or backing of warrants for the data the authors perceived that as an indicator that the particular mathematical idea had become self-evident to the students. Likewise, if a previous claim was used as either data, warrants or backing this was also taken as a sign that the particular mathematical idea represented through the claim had become a mathematical practice. [Stephan & Rasmussen, 2002, p. 462] In this way they were able to document the development of six mathematical practices during the duration of the course.

Ball and Bass share the view that neither the individual nor the sociocultural perspective seems to be sufficient to analyse how mathematical knowledge is constructed in a classroom setting [Ball & Bass, 2000, p. 194]. In order to approach the question of how mathematical knowledge is constructed they propose a highly subject-specific practice-based theory only applicable in *mathematics* classrooms:

We scrutinize classroom mathematics learning and teaching in light of ideas about construction of knowledge that are rooted in mathematics as a discipline. (...) This mathematical perspective makes visible some critical aspects of mathematics teaching and learning that are hidden when viewed from a cognitive or sociocultural perspective. In particular, this analysis allows for and explores a subject-specific view of learning [Ball & Bass, 2000, p. 194 and 195].

In Danish we would name such a view 'fagdidaktisk', and it differs from general learning theories because it uses *mathematical lenses* to view teaching/learning situations in the classroom. Ball and Bass focus on what sort of mathematics is constructed, how students are learning to reason in mathematically accepted ways, and what kind of mathematical resources they make use of in their attempts to convince classmates. The empirical data consists of video- and tape recordings from Ball's mathematics instruction in a third grade class during the school year 1989-90. The teaching experiment, being conducted by only one teacher and in only one class, represents a sort of existence proof, that it is possible to teach students to participate in activities corresponding to those of a professional mathematician. The data presented shows how students are able to make conjectures, to present arguments in favour of or against the conjectures, to

understand the difference between conjectures and definitions, and to understand the purpose of proving.

Blanton, Stylianou and David adopt a sociocultural view on learning when they study how professors use scaffolding in the development of students' understanding of proof production [Blanton et al., 2003]. In sociocultural theory the term *zone of proximal development* is used to indicate a student's developmental potential [Vygotsky, 1978, p. 84]. It is defined as the difference between what a student can do (in a problem solving activity) alone, and what the student might accomplish when assisted by a more knowledgeable peer.

The study is founded on empirical data from a one-year discrete mathematics course at university level where emphasis was placed on mathematical argumentation and proof. As was the case with the study conducted by Ball and Bass, this study also offers an existence proof for the claim: it is possible to engage students in metacognitive discussions. And furthermore, they show that managing to engage students in these kinds of discussions seem to improve the students' abilities to construct proofs.

Instructional scaffolding is one instructional tool that can be applied by teachers or professors. Blanton, Stylianou and David concentrate on the professor's utterances, and categorise the utterances in two groups [Blanton et al., 2003, p. 117]:

- **Transactive prompts** When the professor is asking questions in order to provoke transactive reasoning (criticisms, explanations, justifications, clarifications, and elaboration of ideas) in students the utterances are termed transactive. These prompts initiate transactive discussions among students.
- **Facilitative utterances** In classroom discussions the professor sometimes re-voices or confirms students' suggestions and ideas. These utterances are termed facilitative.

After having analysed the professor's scaffolding in class, the authors turn to analyse students' small group discussions when the professor is absent. The students' types of utterances are now more limited and restricted to a) requests for clarification/elaboration/justification and b) responses to these requests. There did not seem to be any facilitative utterances among students. Where the transactive utterances from the professor were meant to promote transactive discussions, the transactive utterances from the students (or actually from only one of the students) seemed to be a way to negotiate meaning with the other students. Presumably unintentionally, the transactive requests forced the students to clarify ideas, and these requests thus improved their understanding. The conclusion of the study is that whole-class discussions where the professor uses the described kind of scaffolding can advance students' proof construction abilities when working in small groups.

The structures of argumentation during proving processes in junior high school classrooms (six French and German classes) where both teacher and stu-



dents participate are examined by Knipping [2008] using the model by Toulmin [1969] (see page 33). She found two different types of argumentation structures. In a argumentation process with a source structure the process “*arguments and ideas arise from a variety of origins, like water welling up from many springs*” [Knipping, 2008, p. 437]. The process was not explicitly aiming at a final goal, but different ideas and conjectures were instead proposed and examined during the proving process. In a reservoir structure, the arguments “*flow towards intermediate target conclusions that structure the whole argumentation into parts that are distinct and self-contained*” [Knipping, 2008, p. 437]. In classrooms where argumentation processes had a reservoir structure learning to prove was focused on the logical deductions and steps, and claim, data and the need for warrants were explicit. In classrooms where argumentation processes had a source structure focus was on understanding the key idea behind the claim. The act of proving was experienced as a productive, creative process, and theorems and concepts invoked in the process were perceived as interesting in their own right and not just because they were part of a formal argument. Although proving is both a matter of logical deductions and a process of creativity, Knipping’s studies show that teaching ends up separating between the two [Knipping, 2008, p. 438].

Based on observations of one introductory real analysis course, Weber [2004] examines how the professor’s way of presenting certain types of proofs reflects in the way students construct proof. He identifies three different forms that one professor uses when demonstrating proofs in class:

In the *logico-structural* style the professor would do a proof by writing down the definitions of the terms in the statement and the assumptions that he would use. From the definitions and assumptions he would draw inferences leading to the conclusions. The *procedural* lecture style<sup>16</sup> was mainly used for  $\epsilon$ - $\delta$ -proofs. Here the professor would outline the structure of the proof leaving a lot of blank spots where further argumentation was needed, for instance he would write “*Let  $\epsilon > 0$ . Let  $N = \dots$ . If  $n > N$ , then  $\dots$* ”, leaving a blank spot for the expression of  $N$  [Weber, 2004, p. 122]. The needed analysis would be carried out in a separate column and the blank spots would be filled out later. In the *semantic* teaching style the professor would take time to explain or illustrate the statement before proceeding to a formal proof. The study is thus only concerned with proof demonstrations of completed proofs, and does not consider cases where only an outline of the arguments constituting a proof is provided.

When examining six students’ proof productions, Weber found that the students tended to use the same forms as the ones the professor had used.

<sup>16</sup> In form it is similar to the two-column method used in American schools when introducing proofs in geometry [Herbst, 2002b].

### 2.3.2 Enhancing students' conceptions of proof through teaching

Many students in mathematics find the transition between upper secondary school and university very difficult [Moore, 1994]. While upper secondary mathematics is dominated by calculations and algorithms, university mathematics is concerned with establishing the mathematical foundation through axioms, theorems and proofs. Upper secondary students are almost only exposed to proof in geometry, but at university almost all mathematical domains are presented with a high focus on proof. In linear algebra and geometry the students are supposed to construct easy proofs, whereas calculus courses focus more on procedures and algorithms than on proof construction. So although calculus students have been introduced to proofs based on the epsilon-delta/epsilon- $N$  formalism, they have in general not been trained to construct proofs using this formalism.

Furthermore, the presentation form of mathematical textbooks does not make the transition easier. At the advanced level of university mathematics there seem to be a tradition for presenting the material in the form of definition-theorem-proof with examples spread around with a more or less generous hand. This presentation form often comes off on the way the professor reviews the material in class which causes troubles for the students when trying to learn the trade of mathematics:

...the teaching and learning of mathematical proof and proving should not first and foremost focus on students' passive acquisition of readymade proofs constructed by others and presented to them on silver plates. Such an approach is likely to create, amongst students, an image of proof and proving as no more than a particular ritual tribal dance performed only to honour remote gods and goddesses of the mathematical tribe on grounds unknown to mankind. [Niss, 2005, p. 8]

University mathematics students are studying with the hope of becoming professional mathematicians (this both encompasses teachers, pure mathematics researchers, and mathematicians applying mathematics in other fields). In expert practices, justification and proof come after a number of other steps where the researcher uses imagination, intuition and speculation, poses conjectures, readjusts definitions, and finally tries to prove the proposed claim. The process is iterative with many trial and errors. The working process of an expert mathematician is thus both a question of exploration leading to discovery and confirmation (proving), whereas the definition-theorem-proof practice only favours the part of confirmation. In order for students to understand proof and proving it is necessary for professors to distinguish between exploration and proof, and to teach students to explore and discover [Hanna, 2000, p. 14].

In order to make the transition to formal mathematics easier, quite a few transition-to-proof courses or methods where students are supposed to obtain a better understanding of proof and thus improve their abilities to construct proofs have been proposed and studied in the literature [Anderson, 1996].

Teaching practices based on ‘scientific debates’ have as their purpose to teach students the necessity of proof by teaching them to propose and justify conjectures [Alibert & Thomas, 1991]. It was developed in the eighties and has since then been the foundation of a number of teaching experiments conducted at university level [Ruthven, 1989; Legrand, 2001]. An example of how the teacher could initiate a scientific debate is given in [Alibert & Thomas, 1991, p. 226]:

If  $I$  is an interval on the reals,  $a$  is a fixed element of  $I$ , then we set, for  $f$  integrable over  $I$ ,

$$F(x) = \int_a^x f(t)dt.$$

The teacher then asks the question: ‘Can you make some conjectures of the form: if  $f \dots$  then  $F \dots$ ?’

The students would make conjectures such as: “*if  $f$  is increasing then  $F$  is increasing too*” [Alibert & Thomas, 1991, p. 226], and the conjectures would be defended or refuted through class discussion. When asked, students responded that this kind of teaching style made them understand better the questions that the mathematical concepts and results were aiming at answering. The students were involved in the teaching, and took actively part in the discussions.

The Moore Method and the modified Moore Method (MMM) are problem-based approaches to teaching undergraduate mathematics, originally proposed by R. L. Moore in 1949 [Mahavier, 1999; Smith, 2006]. The teaching practice is based on a list of problems which the students are supposed to solve or prove in groups or alone without or with very little assistance of textbooks [Smith, 2006]. In the version of the method used in the study in [Smith, 2006], students were given a set of notes with definitions and theorems to be proven, and some exploratory problems. They were supposed to construct the proofs outside class in groups and then present the proofs before the class and the professor. The lessons would then consist of a mix between proof presentation and discussion of the proofs. The professor would not assess the proof in class but instead leave it up to the class to evaluate and discuss the proof.

Considerations concerning how professors could make students understand proofs made Leron [1983] propose a new way to present proofs – especially proof by contradiction [Leron, 1985] – to students. Since a textbook proof is almost always linear in style it is very tempting for the professor to try to explain the proof by justifying each step in the chain of deductions. This way of presenting a proof is described as “*linear*” [Leron, 1983, p. 174]. In opposition to this style, Leron proposed, and Alibert & Thomas [1991] later advocated, the idea of presenting a proof using a “*structural method*”, which could provide students with a structural understanding of proof. Instead of consisting of successive steps of deductions where the beginning of the proof comes first and the conclusion last the proof is structured in levels. Leron claims that basically all proofs contain a main construction that is the core of the proof [Leron, 1985, p. 323]. He

gives an example with the proof by contradiction of the statement that there are infinitely many primes, where the main construction in the proof is the number  $M = p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$ . The first level contains a description of the main constructions in the proof and contains both the beginning and the end providing the reader with a global view of the proof. After introducing and motivating the main constructions in the proof and global structure of the proof the following levels contain the deductive details. This method will provide the student with an understanding of the bearing idea in the proof, but it does not reveal how or why the main construction in a proof was invented in the first place.

A popular method in American schools for introducing proof is the two-column method where the logical steps/statements in the proof are placed on the left side, whereas the argumentation behind the statements are on the right (similar to the procedural teaching style identified by Weber [2004]). Through an analysis of a proof discussed in a geometry classroom, Herbst [2002a] reached the conclusion that the method is not appropriate for introducing and teaching students the role of proof and how to justify statements. The students do not engage in a process of coming to know mathematics; they participate in a formal game, where they do not come to realise by themselves how proof ideas are invented.

The observation that many university students are able to solve tasks using familiar steps and procedures, but still possess misconceptions and lack conceptual understanding, led Gruenwald [2003] to develop a teaching method, where undergraduate calculus students were provided with false statements to be disproven by counter-examples. The rationale behind this teaching method is that the process of producing counter-examples is a non-procedural activity that demands that the students investigate deeply the properties of the concepts involved in the statement and the proposed relations between them. The method was evaluated through questionnaires where a majority of students reported *“that the method was very effective and made learning mathematics more challenging, interesting and creative”* [Gruenwald, 2003, p. 33] and *“helped them to understand concepts better, prevent mistakes . . . , develop logical and critical thinking, and make their participation in lectures more active”* [Gruenwald, 2003, p. 39].

### 2.3.3 Summary

The literature is sparse when it comes to frameworks applicable for analysing mathematical teaching practices at university level. Researchers tend to single out specific elements of instruction for further analysis, and do not focus on analysing the teaching practices from a more global point of view.

Frameworks for analysing the learning and sense-making in a classroom exist and some of them are also applicable at university level. One such framework focuses on how meaning develops and is shared in the classroom, but it does not provide a means to analyse students' solving learning as it is seen through problem solving.

This section has also presented research reports about experimental and non-experimental teaching practices at university level. These reports show that it is possible to engage students in situations where they learn to reason in mathematically acceptable ways and that problem solving courses can enhance students' conceptual understanding of mathematics.

## 2.4 Justifying mathematical statements in mathematical analysis

Not many research studies take the perspective of proving as a problem solving activity and analyse proving processes [Weber, 2005, p. 352]. Instead, many use task-based interviews to study students' perceptions and beliefs about proofs and proving. In this section justification as a problem solving activity is in focus.

### 2.4.1 Approaches in justification processes

Based on proof tasks from real analysis and abstract algebra Weber [2005] identifies three different approaches students have when trying to produce a proof:

- **Procedural proof production**, where the student uses proofs of similar statements as templates for making new proofs.
- **Syntactic proof production**, where the student lists different definitions and assumptions in relation to the statement to be proved. By use of already established theorems and logical rules, inferences are drawn between these additional statements which are supposed to prove the statement (unless the student gives up).
- **Semantic proof productions**, where the student uses informal/intuitive representation of relevant concepts as a help in producing the formal argumentation. This could for instance be the use of examples to examine the truth of the statement and to base a formal justification on that ground.

There are presumably different learning opportunities attached to the three different proof production approaches [Weber, 2005, p. 358]. Based on a discussion of the different learning opportunities Weber concludes that “*semantic proof productions provide more important learning opportunities than procedural or syntactical proof productions*” [Weber, 2005, p. 358]. In semantic proof productions the student gains conviction about why the statement is true and develops the ability to produce formal proofs based on this conviction. This is basically how a professional mathematician works. But the two other strategies can also provide constructive learning opportunities, according to Weber. Using a procedural strategy the student trains the application of proof procedures connected with the topic, for instance how to make a proof using the epsilon-delta formalism in real analysis. Syntactic strategies train students in making inferences from

definitions and already established theorems, and also acquaint the student with these results.

Gibson [1998] and Alcock & Simpson [2004] study how students use visual reasoning in proof production situations in real analysis. In both studies the students are first term university students in mathematics. From data, Alcock & Simpson [2004] characterise reasoning based on visual imaging to be when 1) the student introduces diagrams, 2) the student makes gestures while explaining arguments, 3) the student prefers to think in terms of pictures or diagrams instead of using algebraic representations, or when 4) the student refers to a sense of meaning derived from a non-algebraic representation [Alcock & Simpson, 2004, p. 9]. They found that the students who could be characterised as visualisers (students who regularly introduce visual images in their reasoning) had the following common characteristics: a) they had a tendency to view mathematical constructs as objects (for instance, they compared sequences), b) they made quick conclusions based on the drawings of what the authors call “*prototypical examples*” [Alcock & Simpson, 2004, p. 12], and c) they were convinced about the truth of the proposed assertions [Alcock & Simpson, 2004, p. 10-14]. Visualisers showed difficulties with the production of written arguments even though they felt that they understood the problem and solution and were able to answer questions about the mathematical material correctly [Alcock & Simpson, 2004, p. 13 and 29].

The use of visual images provided the students with a great deal of confidence but only those students who had an “*internal sense of authority*” were able to seek an integration of visual and algebraic representations, and thus were able to produce mathematically acceptable written justifications. Students with an “*external sense of authority*” regarded written justifications as a form of tribal dance (as described by the quote of Niss [2005] on page 47) without reference to the visual arguments. The authors conclude that the promotion of visual reasoning could have a both positive and negative effect, depending on the students’ beliefs about mathematical justification [Alcock & Simpson, 2004, p. 30].

In contrast with these findings, Gibson [1998] found that the students in his study all started out using verbal or symbolic representations, but shifted strategy and tried to use illustrations instead when the first strategy failed. The use of diagrams served four different purposes: 1) understanding the statement, 2) judging the truth of the statement, 3) discovering ideas, and 4) writing out ideas. There are some overlaps with the findings of Alcock & Simpson [2004], but the striking difference is that the students in [Gibson, 1998] were able to use diagrams to construct formal proofs, which was not the case in [Alcock & Simpson, 2004].

In the study by Smith [2006], proof production strategies of university mathematics students who attended two different courses in number theory, a traditionally taught lecture-based course and a problem-based transition to proof course based on the Modified Moore-Method (MMM) were compared. The study shows

that the two groups of students' solution processes differ on four points [Smith, 2006, p. 81-84]:

- **Use of initial strategy:** In the traditional course, students began searching their memory for a proof technique they could apply, while students in the MMM-course tried to make sense of the statement to be proved.
- **Use of notation:** The traditionally taught students introduced notation with a focus on how they thought they were supposed to prove a statement, while the MMM-students “*introduce notation in logical and natural ways in the context of making sense of the proposition to be proved*” [Smith, 2006, p. 82].
- **Use of prior knowledge and experiences:** The traditionally taught students related the proposition to be proved on prior knowledge and experiences based on surface features, while the MMM-students made the selection based on the concepts involved.
- **Use of concrete examples:** The traditionally taught students did hardly at all use examples in the proof production process, while the opposite was the case for the MMM-students. They used examples partly to make sense of the problem, and partly as a general problem solving strategy.

In order to account for the differences between undergraduates' and professors' proof production processes, Raman [2003] suggests “*that there are three essentially different kinds of ideas involved in the production and evaluation of a proof*” [Raman, 2003, p. 322]. When faced with a statement to be proven, a student might try to gain a personal sense of understanding that the statement is true by looking at specific instances, or by making sketches of the situation. For instance, when faced with the statement “*the derivative of an even function is odd*”, a student looked at polynomials only involving the variable to even power and argued that the derivative would have the variable only to an odd power and therefore be an odd function. The student based the reasoning on – what Raman calls – a *heuristic idea*. Other students might go directly to the definition of the derivative and the definitions of odd and even functions, and try to combine these definitions into a formal proof. They are basing the reasoning on *procedural ideas*, which are founded on formal manipulation with no relation to informal understandings of the concepts involved. According to Raman, procedural ideas do not carry any understanding, but only conviction that the statement is true. This means that heuristic ideas carry personal understanding, while procedural ideas provide the public conviction. [Raman, 2003, p. 322-323]

Raman claims that in order to obtain both understanding and conviction that a statement is true it is necessary to link the heuristic and the procedural ideas. This is done by *key ideas*. The key idea shows why a statement is true. In the example with the derivative of an even function, Raman identifies the key idea as the fact that an even function is symmetrical so that the slope at a point  $x$  is opposite the slope at  $-x$ . The symmetry of an even function explains why the

statement is true and added enough rigour this idea would translate into a formal proof [Raman, 2003, p. 323].

### 2.4.2 Students' difficulties with proof construction

There are many aspects connected to students' difficulties with making justification and constructing formal proofs. From a literature review Moore [1994] lists the following aspects: a) perceptions of the nature of proof, b) logic and methods of proof, c) problem-solving skills, d) mathematical language, and e) concept understanding [Moore, 1994, p. 250]. Beside supporting the finding that students' perceptions of mathematics and mathematical proof could obstruct their proof processes, Moore also found that students' difficulties with proof construction were caused by lack of knowledge of how to use definitions (they were unable to use them in generating new examples, apply them in proofs or use them to point to ways to structure a proof), their concept images and their understanding of notation and mathematical language were inadequate, and they had no idea how to begin a proof [Moore, 1994, p. 251-252, 260-261]. What Moore defined as the students' concept-understanding schemes (concept definition, concept image and concept usage) were thus very insufficient.

Since most university students in mathematics have not taken a 'crash-course' in logical argumentation structures it is not surprising that many students experience difficulties with logical issues of proof. Especially in real analysis, students find it difficult to deal with quantifiers, especially when different quantifiers are combined [Dubinsky, 1997; Dubinsky & Yiparaki, 2000; Epp, 1999], but also the notions of mathematical implication [Dubinsky, 1988; Durand-Guerrier, 2003], and statement negation [Barnard, 1995] cause difficulties. To be able to negate a statement properly is an important tool in proof construction, and a crucial part in indirect proofs (proof by contradiction<sup>17</sup> and proof by contraposition<sup>18</sup>). Often students are confused about the difference between the logical implication of contraposition and the false equivalence  $P \Rightarrow Q \equiv \neg P \Rightarrow \neg Q$  [Epp, 1999; Stylianides et al., 2004].

Also other types of proof strategies cause difficulties among students. Empirical proofs ('proofs' based on specific instances) are often mistaken for real proofs and confused with proof by counter-example [Harel & Sowder, 1998]. As mentioned previously (see page 34), some students find it difficult to accept that a statement can be disproven by only one counter-example, since they think that open general statements are allowed in mathematics [Jahnke, 2008]. A lot of students find indirect proofs harder than direct proof [Epp, 1998; Antonini & Mariotti, 2008], the reason being that they are entering into a false world since what is assumed to be true is actually false and this makes them doubt the va-

<sup>17</sup> Given the statement  $P \Rightarrow Q$  then a proof strategy based on proof by contradiction is to show that  $P \wedge \neg Q \Rightarrow C$ , where  $C$  stands for contradiction.

<sup>18</sup> Given the statement  $P \Rightarrow Q$  then a proof strategy based on proof by contraposition is to show that  $\neg Q \Rightarrow \neg P$ .



lidity of the mathematical rules that normally apply in ‘the real world’ [Leron, 1985; Antonini & Mariotti, 2007, 2008]. In their study Antonini & Mariotti [2008] found that a student was confused whether or not she could assume that  $0/b = 0$ , for  $b \neq 0$  in what she called “*the absurd world*” [Antonini & Mariotti, 2008, p. 406-407]. The study also showed that although a student found it straight forward to formulate the “*secondary statement*”<sup>19</sup> the student had difficulties understanding that the proof of the secondary statement provided him with a proof of the principal statement [Antonini & Mariotti, 2008, p. 407-408]. Many students find it difficult to understand why mathematical induction works as a proof since they feel that the proof relies on a hypothesis (the induction hypothesis) which they have to *assume* is true and thus they cannot be sure about its validity [Ernest, 1984, p. 173][Fischbein & Engel, 1989, p. 281-282].

In a study comparing undergraduate students’ and doctoral students’ abilities to construct proofs for statements in abstract algebra, Weber [2001] found that the undergraduates lacked “*strategic knowledge*” and that was the reason behind their proof production failures. So even though the undergraduate students had a sufficient syntactic knowledge base (knowledge about the theorems required to prove the statement) they did not possess the necessary abilities to derive appropriate properties from the theorems, nor did they have a systematic overview of the proof methods used in this mathematical domain. Weber calls this a lack of strategic knowledge, and adds that this kind of knowledge is heuristics [Weber, 2001, p. 116].

This finding corresponds to a view held by Moore [1994]. He notes that students’ difficulties with proof production not only are due to lack of logical and conceptual knowledge, but also related to insufficient problem solving skills.

So students’ difficulties with proof construction seem to be rooted in lack of knowledge and understanding of various proof techniques, difficulties with identifying the key ideas in proof, and a lack of heuristic ideas for proof construction. Hemmi [2008] views all these issues as a consequence of “*the problem of transparency in the teaching of proof*” [Hemmi, 2008, p. 416]. The notion of transparency originates from theories about social practices developed by Lave and Wenger (see e.g. [Lave & Wenger, 1991]), and is related to the use of artefacts as mediator of knowledge [Hemmi, 2008]:

According to the theory of Lave and Wenger (1991), there is an intrinsic balance in the teaching between the use of artefacts on the one hand, and how to focus on artefacts as such, on the other hand. They call it the condition of transparency. [Hemmi, 2008, p. 413]

In her doctoral thesis, Hemmi advocates the idea of viewing proofs as an artefact [Hemmi, 2006]. Assuming that proof can be viewed as an artefact the above

<sup>19</sup> The statement to be proven is called the principal statement while the statement that is proven in the indirect proof is called the secondary statement [Antonini & Mariotti, 2008, p. 404].

quote says that in a teaching/learning situation involving proof there has to be a balance between focusing on the use of proof (such as the mathematical results and methods provided by proof) and focusing directly on proof (examining proof structure and proof strategies, discussing the role of proof in mathematics and in the history of mathematics, and learning to construct proofs). Both aspects – the ‘invisible’ and ‘visible’ – of proof are needed and support each other when students learn mathematics at an advanced level. This dichotomy within a learning process between seeing and seeing through the artefact is called the condition of transparency. [Hemmi, 2008, p. 414-415] Through her studies, Hemmi found that the problem of transparency could be used to analyse the complexity in the learning difficulties students experience in connection with proof, and that some of the difficulties could be explained by an unbalance between the invisible and visible aspects of the introduction to and handling of proof in relation to teaching and learning [Hemmi, 2008, p. 417-425].

### 2.4.3 Summary

The literature survey shows that students at university level experience many difficulties with the production of mathematical proof. The difficulties have different causes, such as lack of understanding of the role of proof, and of specific proof methods, lack of logical knowledge, and of heuristic knowledge.

As far as I know, the latter aspect has not been paid much attention to in the proof literature. As mentioned, some researchers have proposed methods for proof demonstrations that could make students more aware of the key ideas or main constructions in a proof. But there is a long way from being able to identify a key idea in a proof to being able to produce a key idea in a proof by oneself. Thus, there seems to be a need for research focusing on the use of heuristics in proof production situations.

In general problem solving, heuristics plays a more central role in the research literature, as will be clear in the next section.

## 2.5 Students' problem solving strategies in mathematical analysis

Problem solving strategies in mathematical analysis at university level is closely connected to proof production strategies since the problems to be solved are proof tasks typically or tasks where the solution contains a proof. Research in proof production is often concerned with the ways students reason and justify statements, which has already been dealt with in the previous section. Here I focus more on what has been said in the literature concerning solving strategies, and less on reasoning patterns or students' reasoning difficulties, although studies dealing with non-proof tasks will be included here.

Schoenfeld asked in 1982: “*why do research in problem solving?*” [Schoenfeld,

1982, p. 32]. From his point of view studying students' problem solving skills is a way to learn how students think. And when researchers know more about how students think it may reveal how students should be taught in order to learn to think more successfully, that is in ways that would make them more able to solve problems.

During the 1980s, mathematical problem solving got a central place in the teaching of mathematics and in the educational research literature [Schoenfeld, 1992]. The interest in problem solving, especially regarding problem solving heuristics and solving strategies, was heavily influenced by the famous book of Polya *How to solve it* from 1945, [Schoenfeld, 1992, p. 352]. Polya [1957] presents tricks and ideas that a problem solver can use when solving problems and tasks in mathematics, with special focus on problems in geometry. Polya talks about four phases that the solver has to go through: 1) understanding the problem, 2) making a plan, 3) carrying out the plan and 4) looking back, which includes reviewing and discussing the solution. Besides identifying these phases, Polya presents a list of heuristic ideas to help students when they are stuck in a solution attempt. This is for instance ideas as “*draw a figure*” or help-questions such as “*did you use the whole hypothesis?*”.

But implementation of such heuristics in teaching practices only seemed to have little if any effect on students' success in problem solving [Schoenfeld, 1992, p. 352-353]. The ability to choose the right heuristic process efficiently and being able to carrying it out were some of the major obstacles [Schoenfeld, 1984, p.431]. The lack of implementational success and a proper theoretical basis for studying problem solving from a scientific point of view were presumably responsible for the decrease in research interest that was seen during the nineties [Lesh & Zawojewski, 2007, p. 763]. Although the interest in problem solving, measured by the number of research papers on the subject, is still low some researchers believe that a renewed interest in researching problem solving is on the way, encouraged by the emergence of new research perspectives [Lesh & Zawojewski, 2007, p. 763-764].

### 2.5.1 Definition of problem solving

Solving mathematical tasks has always been an important part of the teaching and learning of mathematics. They have been used as “*vehicles of instruction, as means of practice, and as yardsticks for the acquisition of mathematical skills*” [Schoenfeld, 1992, p. 337]. Often, only a small part of the tasks students are supposed to solve challenge the students to use the mathematical material in new ways. Often, the tasks are training exercises with the purpose to train students in precisely those mathematical procedures, algorithms and techniques just presented in the lectures. [Schoenfeld, 1992, p. 337]

The research area on problem solving is not well defined. It includes studies on how students solve tasks (both problems and exercises), in what ways they reason, verify and justify solutions, and what strategies and heuristics they use

[Southwell, 2004]. The lack of well-defineness of the research area reflects in the lack of an applicable definition of a *problem*. Below are some examples of explicit definitions found in the literature:

- The word *problem* is used ... as a task that is difficult for the individual who is trying to solve it. Moreover, the difficulty should be an intellectual impasse rather than a computational one. [Schoenfeld, 1985, p. 74]
- ... a problem is a mathematical task with no obvious solution or path to the solution and which involves engagement on the part of the solver. [Southwell, 2004, p. 3]
- A mathematical problem is a task in which it is not clear to the individual which mathematical actions should be applied. [Weber, 2005, p. 351-352]
- A task, or goal-oriented activity, becomes a problem (or problematic) when the 'problem solver' (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation. [Lesh & Zawojewski, 2007, p. 782]
- A cognitively non-trivial problem is one where the solver does not begin knowing a method of solution. [Selden et al., 1989]

Another way to characterise a mathematical problem is by stating what it is not. So Polya [1945] defines that "*a problem is a 'routine problem' if it can be solved either by substituting special data into a formally solved general problem, or by following step by step, without any trace of originality, some well-worn conspicuous example*" [Polya, 1945, p. 171] and Schoenfeld defines an exercise as "*... if one has already access to a solution schema for a mathematical task, that task is an exercise and not a problem*" [Schoenfeld, 1985, p. 74].

Selden et al. [1989] propose that the characterisation of a task as either a problem or an exercise depends on both the task and the person who tries to solve it. Beside, that the definition of a mathematical problem must be relative to the person who tries to solve it, Hughes et al. [2006] added another dimension, namely the issue of time. It is obvious that a task or a question that once was a problem can become an exercise, but Hughes et al. [2006] found that what was once an exercise might after a while be transformed into a problem. The possibility of transformation relies on the activeness of a person's "*web of meaning*" [Hughes et al., 2006, p. 95]. If a person is forced to activate or recreate (lost) meanings of mathematical concepts and procedures (through solving what was once an exercise, e.g.) that person is more capable of solving an unfamiliar mathematical problem.

Lester Jr. & Kehle [2003] view mathematical problem solving as an activity with the aim of resolving an emerged tension:

Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity ...that prompted the original problem-solving activity. [Lester Jr. & Kehle, 2003, p. 510]

Mamona-Downs & Downs [2005] propose that the act of reading a proof also can be viewed as a problem solving activity: “.. *it may be worth suggesting to students that reading [a proof] can constitute a true problem-solving activity; it can be as much a challenge to understand a text as it is to manufacture a strategy resolving a given task.*” [Mamona-Downs & Downs, 2005, p. 396].

In the recently published Handbook, Lesh and Zawojewski propose a definition of problem solving inspired by mathematical modelling:

... *problem solving* is defined as the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing and revising mathematical interpretations - and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics. [Lesh & Zawojewski, 2007, p. 782]

This definition of problem solving is part of a new perspective on problem solving presented in [Lesh & Doerr, 2003], *the models-and-modeling perspective*. In traditional perspectives on problem solving, modelling or applied problem solving is believed to be a subset of traditional problem solving, while in the *models-and-modeling perspective* the case is reversed: here traditional problem solving is treated as a subset of applied problem solving (i.e., model-eliciting activity) [Lesh & Zawojewski, 2007, p. 783]. The hope and belief is that this perspective will provide new opportunities for theory building and methodological developments within the problem solving research area [Lesh & Zawojewski, 2007, p. 779-780].

### 2.5.2 Problem solving behaviour

During ‘the Golden Age’ of problem solving research, Schoenfeld published his famous book on problem solving *Mathematical Problem Solving*, founded on empirical data from his own problem solving course for undergraduate mathematics students [Schoenfeld, 1985]. The mathematical problems originated from different fields of mathematics, and were not proof tasks. Schoenfeld used the data to study and characterise problem solving behaviour. He identified four different aspects influencing students’ problem solution processes [Schoenfeld, 1985, p. 15]:

- **Resources** Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand.
- **Heuristics** Strategies and techniques for making progress on unfamiliar or nonstandard problems; rules of thumb for effective problem solving.

- **Control** Global decisions regarding the selection and implementation of resources and strategies.
- **Belief Systems** One's 'mathematical world view', the set of (not necessarily) determinants of an individual's behaviour.

This division is used in the following to structure some of the research findings presented in the mathematics education literature concerning problem solving.

### 2.5.2.1 Resources

It has not been possible to find papers explicitly dealing with the relation between a student's resources and successful problem solving skills. However, it has been possible to find studies comparing students' solving abilities in routine and non-routine problems. Routine tasks might give indications of the sort of resources possessed by a student, or at least this is how they are used in the papers.

In [Wood et al., 2002] 85 linear algebra undergraduate students' results on an written test composed of three groups of tasks were considered. Group A tasks required factual knowledge and routine use of procedures, group B tasks demanded information transfer and ability to make applications in new situations (e.g. to transform knowledge of a routine skill to meta-knowledge of skill explanation), while task solutions in group C included justification, interpretation and evaluation. The marks were distributed such that 88 marks could be received solving group A tasks, 15 marks in group B, and 27 marks in group C. The findings showed that students who did well on the test scored well in all three groups, (a good score might have been obtained by just solving group A tasks) and among those with a bad score the performance was low in all three groups. The study showed that it was possible to have a "*deep learning*" [Wood et al., 2002] without being able to solve routine tasks (being able to solve B and C tasks, but not A tasks).

In a study of 19 above average calculus students' abilities to solve non-routine calculus problems, Selden et al. [1994] found that the students possessed the necessary basic calculus skills, assessed through routine tasks, but that 12 of the students failed to solve any non-routine problems correctly (five problems were posed). Teaching traditional calculus does apparently not make students able to solve problems demanding that they combine known techniques and concepts in new ways. This underpins Schoenfeld's claim that possessing the necessary resources does not guarantee problem solving success. But in accordance with the findings in [Wood et al., 2002], the study by Selden et al. [1994] also provides empirical evidence that a large basic knowledge base (= high score on routine tests) is not a necessary condition for problem solving success: two of the students with lowest score on the routine test managed to reach a correct solution to one of the non-routine problems.

### 2.5.2.2 Heuristics

For students to be successful in problem solving a collection of solving strategies to activate in a problem solving situation must be possessed. The learning of these strategies should thus be something that teaching addresses. Mamona-Downs & Downs [2004] examine how a teaching sequence composed of three tasks could help undergraduates learn a certain very useful problem solving technique, the construction of a bijection for enumeration task. During the teaching sequence the students were guided at two different stages. The first stage was to get the students to construct a bijection (the tasks could all be solved without using this method), and secondly they had to realise the general techniques in the three particular examples. The result was only partly satisfactory since the students' generalisation was not as clear as the authors had hoped and they felt unsure that the students had acquired the desired technique.

Students often perceive that task solutions provided by the teacher or professor contain or rely on 'tricks' or 'good ideas'. In a study of undergraduate tutors' "*pedagogical awareness*" of their students' conceptual difficulties [Nardi et al., 2005], a tutor explained the difference between 'tricks' and techniques. In his view what could at first be perceived as a trick would, after it had been used several times and in different situations, be viewed as a technique, "*what differentiates a technique from a trick is this transferability*" [Nardi et al., 2005, p. 286]. This means that if students do not come to realise the transferability they will keep on viewing a certain strategy as a trick or a good idea.

An important strategy in a problem solving situation is to re-formulate the problem [Silver, 1994; Cifarelli & Cai, 2005], and in one of Polya's heuristic ideas ("*variation of the problem*") [Polya, 1957, p. 209-214], he advocates the viewpoint that all genuine problem solving demands problem alteration:

*Varying the problem*, we bring in new points, and so we create new contacts, new possibilities of contacting elements relevant to our problem. . . . we cannot hope to solve any worth-while problem without intense concentration. But we are easily tired by intense concentration of our attention upon the same point. In order to keep the attention alive, the object on which it is directed must unceasingly change. [Polya, 1957, p. 210]

Cifarelli & Cai [2005] found that college students when trying to solve open-ended problems<sup>20</sup> managed to have an exploration phase<sup>21</sup> composed of several problem posing and solving attempts, which were used to make sense of the problem situation.

Lithner [2008] proposes a framework for analysing students qualitatively different ways of reasoning during problem solving. Even though the quality of

<sup>20</sup> The authors describe open-ended problem situations as situations where "*some aspect of the task is unspecified and requires that the solver re-formulates the problem statements in order to develop solution activity*" [Cifarelli & Cai, 2005, p. 302].

<sup>21</sup> Apparently not defined as in [Schoenfeld, 1985].

the reasoning is in focus, the framework can also be seen as a categorisation of students' problem solving strategies, generally seen. The framework operates with two major types of reasoning: imitative reasoning (memorised reasoning and algorithmic reasoning) and creative reasoning (creative mathematically founded reasoning). All the categories and sub-categories in the framework are related as follows [Lithner, 2008, p. 258-259, 262-264, 266]:

- **Memorised reasoning** The strategy choice is founded on recalling a complete answer, and the implementation consists only of writing it down.
- **Algorithmic reasoning** The strategy choice is to recall a solution algorithm, and the implementation consists of substituting relevant data from the task.
  - **familiar** The task is seen as being of a familiar type and can be solved by a corresponding known algorithm, and the task is solved by implementation of the algorithm.
  - **delimitting** An algorithm is chosen from a set that is delimited by the reasoner through the algorithm's surface relations to the task. The outcome is not predicted. If the implementation process does not lead to a satisfactory answer the implementation process is terminated without attempts to understand the reason for the failure.
  - **guided**
    - \* **text** The strategy choice concerns identifying surface similarities between the task and an example, definition, theorem, rule, or some other situation in a text source, and the implementation does not contain any verificative argumentation.
    - \* **person** All problematic strategy choices are made by a guide, who provides no predictive argumentation, and the implementation follows the guidance and executes the remaining routine transformations without verificative argumentation.
- **Creative mathematically founded reasoning** A for the reasoner new reasoning sequence is created, or a forgotten one is re-created (condition of novelty). There are arguments supporting the strategy choice and/or strategy implementation motivation why the conclusions are true or plausible (condition of plausibility). The arguments are *anchored* in intrinsic mathematical properties of the components involved in the reasoning (condition of mathematical foundation).

The framework is an extension and refinement of the framework mentioned in the introductory chapter (see page 3) where the categories were reasoning based on established experiences, identification of similarities, and plausible reasoning [Lithner, 2003], and the same data from the previous publications was used as an illustration of this new edition.

There is a resemblance between the strategy where students use similarities between the task and other solved tasks or examples in the textbook, and the *key-word algorithm* described by Schoenfeld: specific words in the problem formulation are used to choose relevant solving strategies [Schoenfeld, 1982, p. 27-29]. Both strategies are superficial in the sense that they do not rely on mathematics



and it is difficult to see that students who use this strategy will gain more mathematical insight, both regarding mathematical concepts and also when it comes to learning to apply mathematical procedures. These methods can be successful as long as the teacher presents the students with types of tasks that they are familiar with and have been trained to solve. The students are able to solve the tasks – using the key word algorithm – and the students and teacher can convince themselves and each other that all is fine and that the mathematics has been well understood. But according to Schoenfeld this is both “*deceptive and fraudulent*” [Schoenfeld, 1982, p. 29].

As mentioned in the introductory chapter, Lithner shows that university students hesitate to use plausible reasoning/creative reasoning even when this kind of strategy might have lead them further towards a (correct) solution. Lithner speculates that the preference for using superficial strategies are “*caused by insufficiencies in the learning environment*” [Lithner, 2008, p. 273] and by the textbook tasks that students solve [Lithner, 2000a, p. 95] [Lithner, 2000b, p. 188] – a view shared by Harel [2008]. The hypothesis is substantiated through a study of the types of reasoning required to solve calculus tasks from a traditional American calculus textbook [Lithner, 2004]. The findings are that 90% of 598 tasks could be solved by identifying and perhaps making small adjustments to a procedure or method in the textbook. Textbooks do thus not promote the use of plausible or creative reasoning.

### 2.5.2.3 Control

Even if a problem solver has access to a complete list of known heuristics it is still necessary for him to make reflections about whether to chose one solving strategy above another, and to assess whether the chosen strategy actually works. The issue of control in a problem solving situation has also been studied in the literature under the term metacognition<sup>22</sup> [Schoenfeld, 1985, p. 137]. To possess metacognitive skills is very important if a student is to succeed in problem solving situations, but these skills are often not acquired by students and only rarely taught by professors in traditional lecture-based teaching practices.

Besides characterising problem solving behaviour Schoenfeld also proposed a way to analyse solution processes. Inspired by Polya, Schoenfeld distinguishes between the following phases in a solving process: reading, exploration, analysis, plan, implementation, and verification [Schoenfeld, 1985, p. 297-301]. Using this division he showed that students or novices in contrast to expert problem solvers went through fewer stages of control where they evaluated their argumentation and solution attempts [Schoenfeld, 1985, p. 293]. The lack of elements of evaluation and control resulted in ‘wild goose chases’<sup>23</sup> and this was the main cause

<sup>22</sup> Beside dealing with regulation of cognition as the case is with control actions during problem solving, the literature also treats issues related to knowledge about cognition as metacognition [Schoenfeld, 1985, p. 138].

<sup>23</sup> Situations where students for instance spend time making a lot of calculations without being

why the students spent too much time solving the problem or even could not reach a solution to the stated problem.

In his studies, Lithner [2000a] also found a lack of monitoring and control in undergraduates solution processes of calculus tasks. The instances of control that were actually present were based on familiarity and not founded on mathematical properties, and for that reason these instances did not lead to solution success.

Based on college students' solutions to routine and non-routine problems in elementary algebra, Lerch [2004] found that students' abilities to make control decisions were highly depended on their mathematical resources and strategies. In routine problems where students' knowledge base was composed of various resources and strategies the students had success with their control decisions, whereas control decisions lacked in solution processes of non-routine problems where the students' knowledge base of strategies was reduced or maybe even empty.

#### 2.5.2.4 Beliefs systems

Studying undergraduate calculus students' beliefs about mathematics through questionnaires, Juter [2005] found that a majority of the students believed that mathematics is a collection of facts, procedures that have to be remembered, and the goal of learning is to become successful problem solvers. Focus is not on theory nor on understanding. These students also answered that "*mathematics is about coming up with new ideas*" [Juter, 2005, p. 100]. Juter found this statement contradictory to their others more shallow beliefs about mathematics since they would be in a better position to develop new ideas if they were more focused on theory and would "*explore the features of the processes and objects on which they are working*" [Juter, 2005, p. 105]. When comparing students' self-confidence (based on acceptance to statements such as "*I usually understand a mathematical idea quickly*" and "*I can connect mathematical ideas that I have learned*" [Juter, 2005, p. 100]) with problem solving success, Juter found that the more confident students performed better on limit problems than the less confident.

The importance of self-confidence and solving history on problem solution processes is studied in [Lerch, 2004] (same study mentioned above). Lerch found that a student's belief system influenced how the student approached a problem. Students with low self-confidence or a solving history without many successes gave up working on the problems for reasons such as not being able to provide a solution sufficiently quickly (according to their own views), or if the wording of the problem reminded them of problems which previously caused difficulties.

The change of students' perception of task difficulty after attempting a solution is the focus of research in [Wood & Smith, 2002]. The study showed that a majority of 70 first semester students rated the conceptually demanding tasks as more difficult than tasks requiring recollection of facts or the use of routine

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aware of what the results of the calculations might be used for.

procedures. In five out of eight tasks they did not change their opinions about the difficulty of the task after attempting a solution (one task was harder than expected<sup>24</sup> and two tasks were found easier).

Using a taxonomy of categorisation, [Craig, 2002] study different factors' influence on 660 first year calculus students' ranking of difficulty of word problems, and compared the findings with the ranking done by 20 experts. Not surprisingly, they found that students ranked familiar and concrete problems as easier than less familiar and abstract problems, and problems with diagrams were perceived as easier than problems without diagrams (which were not the case for experts), where problems containing rectangles were perceived as easier than problems containing circles.

de Hoyos et al. [2002] suggest that observed differences between two undergraduate students' problem solution processes could be explained by the assumption that the students held two different views of mathematical development. One student's activities could be described as *discovering* key ideas, while the other student's activities were centred around the *invention* of key ideas. The authors speculate that a process based on invention might be more flexible than a process based on discovery, and thus more successful in a problem solving situation.

### 2.5.3 Summary

The research area of problem solving is marked by the different meanings attached to problem and problem solving. The studies presented in this section all claim to concern problem solving, and the tasks used in the studies are characterised to be either routine or non-routine problems but none of the studies discuss this characterisation with respect to the student population.

To succeed in a problem solving situation both mathematical knowledge and computational skills are needed, but it is also necessary to possess and be able to use metacognitive skills so that solving strategies chosen and results obtained can be evaluated during the solving process and perhaps be corrected.

The main point that I would like the reader to notice about students' problem solving difficulties is that different researchers seem to be able to identify the same characteristic: many university students use superficial strategies not embedded in the definition of mathematical concepts or objects presented in the problem. This characteristic was also seen in the section about justification processes where the tasks were proof tasks. Some authors speculate that teaching practices and textbooks give rise to such behaviour.

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<sup>24</sup> The authors were uncertain about the reason for this particular shift, and suggested that the complexity of the task was hidden (the tasks was: show that  $x^3 + cx + d = 0$  has only one root if  $c \geq 0$  [Wood & Smith, 2002, in appendix]). Taken into consideration that the students found conceptual tasks more difficult than procedural tasks, it is very reasonable to assume that they characterised it to be procedural at first sight, despite the request to construct a proof, but after trying to solve it they realised that the solution demanded conceptual understanding of 3rd degree polynomials, roots and derivatives.

## 2.6 Returning to the research questions

In this section I discuss the relevance of some of the literature presented in the previous sections for the examination of the research questions.

The research questions stated in the introductory chapter are basically designed to determine how student solving approaches are influenced by the teaching practice the students take part in (assuming that there is such a relation between the two). Trying to find answers to the research question originally intended, my initial (and naive) expectation was to use pre-made frameworks for categorising teaching practices on the one hand and student solution processes on the other hand, as well as to look for any coincidence or convergence between the two categorisations in a concrete case.

At an early stage of the process, I became familiar with Lithner's work on characterising undergraduate students' reasoning processes, and his hypothesis about how the teaching practice might result in the unwillingness of students to use plausible reasoning. The categories looked as if they also could be applied to describe student solving processes when the tasks asked for or the solution should contain a proof. I then began searching for a framework that characterises teaching practices at undergraduate level. After having read section 2.3.1, the reader will know that frameworks for characterising teaching practices or the influence of certain aspects of teaching practices do exist. Some of them are not made with the undergraduate level in mind, but they are sufficiently general in nature to be easily applied at university level (one example of undergraduate level application was actually provided), while others have been developed to describe specific aspects of professors' actions.

The framework of Cobb and co-workers can be used to analyse the establishment of social and sociomathematical norms and mathematical practices in the classroom, also classrooms at university level. These general aspects of teaching practices seem highly relevant to analyse, also with respect to the stated research questions, but it is far from obvious how an analysis of these aspects can be related to or provide an explanation of student solution processes. Thus, this framework is usable in the analysis of parts of the teaching practice, but does not seem to be able to do 'all the work'.

The model of teaching developed by Schoenfeld and his colleagues is designed to explain and predict the actions of teachers and professors in the classroom based on, for example, their articulated lesson image. The research questions focus on the relationship between teaching and student solution processes, while the model focuses only on explaining the actions of professors and teachers. As a result, the model seems to be partially inadequate for analysing the relationship.

The rest of the studies presented in section 2.3.1 are sets of categories for characterising aspects of teaching practices as opposed to actual frameworks. The categorisation of a professor's different styles of proof (the study by Weber [2004]) concerns a specific aspect of a teaching practice, and although this characterisation is the only one that relates aspects of teaching and student problem solutions

to each other, the study only focuses on the result of the solution processes and not on the actual processes.

The analysis of a professor's scaffolding and how it affects student group discussions (the study by Blanton et al. [2003]) seems to provide a set of useful terms for describing what the professor is doing in class discussions during the lectures and when he or she assists the students while they solve tasks in the classroom. But the characterisation is limited to utterances and cannot be used to analyse situations where the students are not engaged in a dialogue, for instance during certain passages of a textbook proof demonstration where only the professor speaks.

Since the research literature does not seem to be able to provide a pre-made framework for characterising teaching practices and relating it to students solution processes, I found it necessary to develop my own approach.

## 3 Methodology

“... in a majority of articles in journals and chapters in books, a description is provided of ‘how’ the research was done but rarely is an analysis given of ‘why’ and, more particularly, out of all the methods that could have been used, what influenced the researcher to choose to do the research in the manner described.” (Leone Burton)

In this section I describe and discuss in more detail the different parts of the empirical design, how they relate to the research questions, and the choices I made regarding classroom observations and observations of students’ solution processes.

### 3.1 The nature of the investigation

From the introductory chapter it should be clear to the reader that I have chosen to base my investigation of the research questions on empirical data. The research questions do not dictate which scientific method would be appropriate in the search for answers. The questions could as well have been addressed from a theoretical perspective, through literature studies or from a historical point of view, for instance. Since the research field of mathematics education, and particular mathematics education at the tertiary level, is fairly young, it seems that there is a need for developing an empirical base of knowledge on which theoretical studies can emanate. The age of the field also entails that the body of scientific literature is limited, which would complicate the execution of literature or historical studies.

#### 3.1.1 Choice of empirical methods

There is an ongoing discussion about the purpose of dividing empirical methods into qualitative and quantitative methods and a discussion concerning the definition of the two parts [Mason, 2002, p. 2-3]. Mason [2002] gives the following characteristics of qualitative research in social science:

Qualitative research is

- ...concerned with how the social world is interpreted, understood, experienced or produced.

- based on methods of data generation which are flexible and sensitive to the social context in which data are produced (rather than rigidly standardized or structured, or removed from 'real life' or 'natural' social context, as in some forms of experimental method).
- based on methods of analysis and explanation building which involve understandings of complexity, detail and context. Qualitative research aims to produce rounded understandings on the basis of rich, contextual, and detailed data. (...) Qualitative research usually does use some form of quantification, but statistical forms of analysis are not seen as central. [Mason, 2002, p. 3-4]

The goal of my project is to investigate the link between university teaching practices and the ways students solve proof-related tasks. The two main research questions (see page 4) are methodological related in the sense that a natural research strategy would be first to examine students' solving difficulties, and then observe the teaching practice and analyse data from the perspective of the identified difficulties. This is not possible in practice, since the teaching practice has ended before the students' solution processes can be examined (they have to learn the subject matter before they can solve tasks). Therefore, it is necessary to observe the teaching practice and compose data material *before* the nature of the solving difficulties has been examined. A reasonable research strategy would be to observe and characterise the teaching practice independently of a characterisation of the students' solution processes. In order to answer the second research question a way to characterise teaching practices must be developed and applied after the students' solving difficulties have been examined.

In order to characterise both the teaching practice and the students' solution processes, it is necessary to observe the teaching and observe the students when they try to solve tasks. So the methods must include observations of the teaching practice and the students' task solving abilities. Since the literature survey revealed a lack of pre-made frameworks for characterising teaching practices, it is reasonable that the investigation at an initial stage concerns the development of a way to do that. This implies that a thorough investigation of single cases must be the first step and more quantitative methods (investigating several teaching practices in order to apply statistical methods) must come later. That does not mean, however, that some aspects of quantification can not enter the investigation of a single case. For instance, it could make sense to count how many times the professor poses questions to the students during class or how many times he chooses to present solutions to assigned tasks on the board during the course.

What type of data would it be appropriate to generate? The primary data has been chosen to be observations of teaching practices and students' processes of making justifications, but this choice is not dictated by the research questions. Many research studies of a social nature rely on questionnaires or interviews as primary data. Questionnaires and interviews can be designed rather differently, but essentially they are used in order to capture the *opinions* or *experiences* from the research subjects on questions or themes related to the research questions.

I have chosen that students' and professors' opinions are not the main focus, but nevertheless this kind of information could illuminate the problem area from another perspective. If I was interested in using the students' and professors' opinions and experiences as primary data it would have been possible to found the investigation on quantitative data, for instance in the form of answers to standardised questionnaires.

### 3.1.2 Methods for data generation in the classroom

What kinds of tools are available for data generation in a classroom and what are the advantages and disadvantages of these tools? Associated with classroom observations (and observations of students solving tasks), video or sound recording has, according to the literature, been the main tools for capturing 'reality'. But every time a researcher enters a classroom with observational purposes (or observes students working with task solving), the generated data – using one tool or the other – will always only provide an edited segment of that reality. This is why I, as suggested in [Mason, 2002, p. 52], do not speak of 'data collection', but instead use 'data generation' or 'data production'.

Making video recordings of classroom teaching is not without its problems. Entering the classroom as the only researcher, it is practically impossible to operate more than one camera. In a traditional classroom where the students sit in front of the professor and the blackboard, it is not within reach for one camera to record the blackboard and the faces of the students and the professor at the same time. The researcher has to choose a preferred recording direction.

Sound recording has the advantage of being more invisible and less disturbing to the subjects being observed, but has the obvious disadvantage of not being able to record anything else but the sound. Body language, face expressions, direction of attention, writings on the blackboard and so forth are not captured and practical problems such as transcription difficulties can occur if the researcher does not have enough familiarity with the subjects to differentiate between voices.

Logs and research diaries<sup>1</sup> have the disadvantage of depending on the researcher's state of mind at the particular time and are very subjective representations of reality. If the researcher needs to consult data in order to check new insights and hypotheses developed during the data generation as the case is with for instance 'the constant comparative method' used in Grounded Theory [Strauss & Corbin, 1990], logs and research diaries are insufficient.

Then there is the question of whether the classroom observations should be non-participant or participant observations. When the researcher enters the classroom and behaves like 'a fly on the wall' this behaviour is called non-participant observation, because the observer does not act as a participant (for instance, as a professor or a student) [Bryman, 2001]. Participant observations is a term

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<sup>1</sup> I define diaries as records of the researcher's observations and interpretations while logs are (more) objective records of aspects of the research process.



used for “*undercover investigations*” [Cohen & Manion, 1994, p. 107]. Here the researcher identifies herself with the objects of her research, tries to blend in, and behaves like them – with or without their knowledge. This method requires that the research only concerns one group of people. This is for instance not the case in the present research project where both the professor and the students are important categories of research objects. Using participant observations as a research tool, the researcher cannot use any form of recording instrument during the observations, nor write notes. Furthermore, this method requires some serious ethical considerations, not to speak of considerations regarding data validation [Cohen & Manion, 1994, p. 111]. On the other hand, choosing participant observations instead of non-participants observations the research objects will be less disturbed since they do not know (or quickly forget) that they are being observed.

When making longitudinal studies of classrooms, for instance, it is impossible for one researcher to transcribe all the material and have full overview of all the events and episodes. Some sort of records or journals of the data have to be made during the data generation and that will inevitably introduce a manipulation of the ‘raw’ data. Cohen and Manion describe non-participants observations as observations that are founded on a set of observational categories [Cohen & Manion, 1994, p. 109], but non-participants observation can also be done without a pre-structured set of categories, which is the case with the observational methods used in relation to the grounded theory approach [Strauss & Corbin, 1990].

### 3.1.2.1 Secondary data generation in the classroom

I stated above that my primary interest is not to make a characterisation of the teaching practice based on the professor’s or the students’ views and beliefs which is why the classroom observations are seen as primary data. But nonetheless, there are some benefits in examining the views of the parties involved. First of all, the students and the professor are essential parts of the teaching environment and their views on and beliefs about the teaching practice are for that reason interesting and important. Secondly, an inquiry with the purpose of getting the involved parties’ perception of the teaching practice would also provide data to be used in data validation.

An obvious way to get students’ and the professor’s opinions is of course to ask them, but this can be done in many ways. In my view, there is a continuous transition between structured research interviews at one end and completely structureless ‘small-talk’ conversations at the other. In structured interviews a set of questions is posed to the interviewee. The order of questions is not changed from interviewee to interviewee, and the interviewer does not pose any additional questions. This is one pole. This form of interviewing is in a way similar to a questionnaire, where the interviewee instead of given written answers provides the researcher with oral answers which allows a higher degree of elaboration. A

drawback is of course that it is time-consuming, and limiting with respect to the number of interviewees compared to questionnaires.

At the other end of the scale, the completely structureless conversations are placed. If the structureless conversations take place as part of a research study, it is natural that the researcher has an idea about the topic he or she wants to explore. But no specific questions are prepared in advance. This type of interviews is likely to be used in extremely exploratory studies as a source of inspiration, and a drawback is of course that it is difficult to compare the responses of two different interviewees.

In between these two poles there are a number of different variants of semi-structured interviews, for instance interviews based on a set of questions prepared in advance where the order is maintained but where the interviewer is allowed to pose clarifying questions. Another variant could be that the researcher is allowed to pursue further any interesting point the interviewee might bring up during the interview, but returns to the order of questions when the particular point of interest has been explored.

The semi-structured interview is in that respect a tool that allows the researcher to get fairly comparable answers from the group of interviewees to the same set of questions, but at the same time reduces unintelligible answers and misinterpretations.

### 3.1.3 Methods for data generation related to task solution processes

The first research question concerns students' solving processes. The literature is full of studies of students' task solving abilities, at many different educational levels, using different methods, and with different purposes. Research studies using task-based interviews, where individual students are asked to solve one or several tasks while 'thinking aloud' is based on the assumption that there is a correlation between what is going on in the student's head and what is coming out of his mouth [Ericsson & Simon, 1993]. In such studies the researcher often asks the student questions during the solution process to find out "*what the student is thinking*" [Weber, 2001, p. 104]. In other studies the student is also interviewed after the process has been analysed by the researcher to provide data for validation of the data analysis [Lithner, 2000a; Raman, 2002]. The classical study of students' problem solving abilities conducted by Schoenfeld examines pairs of students, while they are working on a problem [Schoenfeld, 1985]. Schoenfeld discusses the arguments for grouping students and not conducting the investigation on individual students. He finds that students tend to be more relaxed when working in pairs instead of alone, and that group work in a more natural way will lead to conversation [Schoenfeld, 1985, p. 281-282].

The purposes of using task based interviews or task based observations (where the researcher does not interrupt) differ. Some researchers use task based interviews to examine students task solving abilities or difficulties [Weber, 2001; Lithner, 2003], while other studies use these methods as more indirect tools for

investigating for instance students' beliefs about proofs [Coe & Ruthven, 1994; Raman, 2003].

Many researchers have used written answers to tasks for analysing students' abilities and difficulties with task solving (see e.g. [Selden & Selden, 1995; Anderson, 1996; Dreyfus, 1999; Segal, 2000; Hoyles & Küchemann, 2002; Stylianides et al., 2004]). Wanting to investigate solution processes it is necessary to have access to the processes and not only the results of processes such as the (often written) solution to a task. Therefore, it seems difficult to base a study of students' task solution processes and strategies solely on written answers produced in the absence of the researcher (or a recording device). It is likely, however, that written answers to tasks could be used to shed some light on the research questions. Two options are available: Either the researcher persuades students to hand in answers to one or a set of tasks or the researcher gets permission to obtain students' answers to tasks that they are obliged to solve in order to pass the course. This could for instance be weekly assignments or tasks from the exam. The first option could be difficult to carry out as it might be too much to ask for. University students do not normally have a lot of spare time to use for solving extra tasks. The second option implies that the tasks are chosen by the professor and his or her choice does not necessarily match the kind of tasks the researcher wishes the students' response to.

## 3.2 Research design of the pilot study

The course investigated in the pilot study was an advanced mathematical analysis course. The subject matter was continuity, differentiability and integrability (Lebesgue integral) for functions of several variables in the abstract frame of metric spaces. Twelve students were enrolled in the course, but only eleven took the final exam. The exam consisted of a three days 'take-home' examination, where the students had to solve four tasks with subtasks. Two weeks later each student should defend his or her answers in an oral examination. The exam had an external examiner<sup>2</sup> together with the professor and was graded. The class met twice a week for three hours during twelve weeks.

### 3.2.1 Classroom observations

The classroom observations were chosen to be non-participant observations. The method of participant observations was ruled out for several reasons: 1) I would not have access to verbatim data records of what had been said in the classroom, and without these an analysis of how mathematical analysis was taught in the classroom seemed impossible; 2) Obviously, I would only have the choice as to

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<sup>2</sup> In Denmark, it is tradition that both the professor and an external examiner evaluate the students at final exams (both oral and written) in the final year of secondary school, at some exams in upper secondary school and at university.

act as a student, and this would possibly impose a bias towards this group and reduce the objective observation of the classroom; 3) and a practical reason was that since I had met the students before, it would have been impossible to act ‘undercover’. I found that these disadvantages by far exceeded the advantage of using participant observation, namely, that the students’ activity would not be influenced by the presence of an observer.

Classroom observations were structured around a set of pre-developed categories. The categories were placed as one dimension in an observation matrix with time as the other dimension. The three hour lessons were cut up into ten minute intervals. An example is shown in figure 3.1, where four of the categories can be seen.

The observation template was constructed in order to study mathematics teaching taking place in a classroom and under the management of a professor. It can not be used to analyse other kinds of teaching situations, such as supervisor-guidance in project work, or the kind of teaching that takes place among students in a study group. During the data generating process the categories were adjusted if some aspects of the teaching practice did not fit into any of the categories. The following categories were used in the pilot study:

- Teaching/learning activities
- Task solving activities
- Extra-instructional activities
  
- Mathematical content
- Illustrations on the board
- Mathematical techniques
- Reference to mathematics worked on in earlier courses or previously in the given course
- Putting things in a perspective/historical remarks
- Anecdotes/detours
- Navigation/presentation of agenda, motivation of results
  
- Students’ questions
- Students’ reactions
- Professor’s questions, comments and reactions

The first three categories describe which activities the professor and students are engaged in. While the investigation aims at detecting a relationship between students’ task solving strategies and processes, and elements in the teaching practice task solving has a separate category. During the course it sometimes happened that the professor discussed the structure of the course, the amount of homework that he expected the students to do, the overall purpose of the course and so on. These episodes are filed under *Extra-instructional activities*. The fourth category is for keeping track of what mathematics the professor or the students are talking

about. Combining this category with the textbook content makes it possible to obtain knowledge about the role of the textbook in the lectures.

Observationsskema til forelæsningsdelen (torsdag)      Dato: 3/3-05      Nr.: \_\_\_\_\_

Tidspunkt ----- Antal studerende	Spørgsmål fra studerende	Undervisningsaktiviteter (enetales, dialog, spørgsmål, opgaveregning, pause)	Opgaveregning (fælles ved tavlen, i grupper, med sidemanden, alene)	Udenomsaktiviteter (informationer, ex. om pausens længde, disciplinering)
9.00-9.10		Snale om strukturer indholdet af resten af kurset		L vil gerne spørge omvendt over (kapitel 11.7) L vil gerne have om mål og integration Snale om Rodins bog Snale om resten af kursets indhold
09.10- 09.20	De studerende efterspørger overskuelighed (hvert fald indirekte)	Indledende snale om kapitlet. L taler overordnet om sædvanen 11.41		
09.20- 09.30	Hvordan: spørges om L, hvor stærket forrest: $Df(f'(y)) \circ Df'(y) = Id$ ②	L taler, de studerende stiller forbehold om sp. til evt. fejl på tavlen.		
09.30- 09.40	Snale: karakteristisk i forhold til PR=11.11 eksempel.	L taler		

**Figure 3.1** An example of the observation template in Danish. The four categories displayed here are from the left *Students' questions*, *Teaching/learning activities*, *Task solving activities*, and *Extra-instructional activities*.

The next six categories contain information about the teaching instruments the professor uses. *Illustrations on the board* contains pictures, diagrams and templates of all kinds that the professor or students use to illustrate mathematical issues, proofs or conceptual relationships. The next category termed *Mathematical techniques* includes situations where the professor explicitly focuses on mathematical techniques which include discussion or mentioning of proof techniques, for instance direct proof or proof by induction. The categories *Reference to mathematics worked on in earlier courses or previously in the given course*, *Putting things in a perspective/historical remarks* and *Anecdotes/detours* contain situations where the professor during lectures or task solving sessions mentions or discusses concepts, theorems or proofs techniques which they have talked about earlier or have encountered in a previous course, situations where the professor puts the mathematics into perspective for instance or if the professor uses 'stories' to illustrate points or explain concepts by, respectively. The category *Navigation/presentation of agenda and motivation of results* contains situations where the professor explains why he talks about a certain concept, why he presents or talks about a theorem, proof or example, and also situations where he explains

the purpose of the course or specific parts of it (for instance the purpose of having solving sessions).

The last three categories are for keeping track of those questions that can not be placed in any of the other categories. Emotional reactions such as frustrations, despondent attitudes, expression of aha!-experiences, students having difficulties understanding the professor or vice versa are filed here.

The observation template made it possible to carry out parts of the analysis of the teaching practice during the observations. Categories were adjusted in order to provide a well-covering picture of the teaching practice.

### 3.2.2 Semi-structured interviews

The classroom observations provided ideas for the semi-structured interviews. Two sets of interviews with the students and the professor were carried out. One set half way through and one set just after the final exam. In each set of student interviews the same collection of questions was posed to all the students, but the interviews were structured as a conversation so the questions did not necessarily come in the same order and additional questions were asked if a student touched upon something of seeming importance. Small-talk was also allowed in order to make the interviewee feel comfortable.

Even though the choice to use semi-structured interviews made it more difficult to compare the students' responses and opened up for the possibility that new questions introduced through the interviews had not been posed to all the students, I still found that the advantage of having more elaborate answers and the possibility to ask the interviewee to clarify answers exceeded the disadvantage. Also, since the number of students was relative small, waiving the possibility of having quantitative results did not seem as a great loss.

English translations of all the interview questions in the pilot study are listed in appendix G.1. All the interviews were audio-recorded and then transcribed.

#### 3.2.2.1 The first set of interviews

The intentions with the first set of professor-interviews were mainly to get information about the professor's intentions with the course and his expectations of the students' classroom activity and their study habits. The student-interviews aimed at obtaining information about their perception of the teaching practice, their own role in the teaching and their opinions about the aim of solving mathematical tasks.

#### 3.2.2.2 The second set of interviews

In the second set of interviews, attention was paid to the students' and the professor's opinions about the conversation or dialogue during lectures. In the evaluation of the course the professor had stressed that there had been a lack of dialogue or debate during the lectures and he had asked the students how he could have persuaded them to participate more actively in the teaching.

One of the interview questions for the students was how they had interpreted the professor's statements about the dialogue and their own participation in the lectures. Other questions concerned the final exam, the students' study habits and the study habits that they thought the professor expected or found optimal.

The second interview with the professor was more loosely structured than the first interview had been. It was focused on what he actually meant by 'dialogue' and if it, according to him, could be possible to have a dialogue given the way the students prepared for class (information obtained from student interviews) and from his own recommendation of adequate preparation routines.

### 3.2.3 Response to interpretations of observations

At the final lesson of the course the professor and students evaluated the course. After the evaluation I presented my interpretation of what I had observed in the classroom and of the first set of interviews. The students gave their response. I invited the professor to participate, but he did not make any comments at that time.

The purpose of this presentation was of course to ask the students if they shared my interpretations of what had happened in the course as a way to examine the validity of my analysis. If the students had disagreed with my interpretations that would not have been enough to dismiss my conclusions. If they had disagreed I could have gone back to look at my data again and analysed it with the students' remarks in mind.

The evaluation and the response to my interpretations were also used as data to characterise the teaching practice as well as the students' views on it.

### 3.2.4 Constructed task solving sessions

The constructed solving sessions were, as mentioned, placed after the last course session but before the written exam. I told the students in advance that the content of the tasks was within the course curriculum. English translations of the four tasks are listed in appendix D.2.

Four tasks were chosen to provide a fair representation of the content of the course. They were designed to test the students' conceptual understanding of the various concepts introduced in the course. The tasks concerned continuity in metric spaces, differentiability of functions of several variables, sigma-algebras and Lebesgue integrability of one-variable functions. It was not the idea to test the students' abilities to perform difficult symbolic manipulations or complicated calculations so the examples of functions and spaces used in the tasks were made as simple as possible and deviated in that respect from examples and tasks used in the textbook.

Eight students volunteered and four teams were made. In three of the four teams it was possible to group students who normally worked together. Since students who normally work together tend to be on the same mathematical level,

the division strategy entailed that this was the case for three of the four teams. In the fourth team I had to pair two students on very different levels due to illness of a student who had originally agreed to participate.

The students got all four tasks at the same time and they were told that the session was planned to take a couple of hours. They were also told that I probably would stop them during the attempt to solve a task and ask them to move on to another task because of time limitations. This interruption would not necessarily mean that they were on the wrong track or that a solution was out of reach. It could also be because I got the feeling that the solution from then on was straightforward for the students.

The four constructed solving sessions were audio-taped and I made notes during the observation of the students' silent actions (for instance if they looked in the textbook) and of ideas for the analysis. I abandoned the idea to use video-recordings because of the possible pressure that video-recordings could put on the students. I afterwards regretted that decision because a lot of important information was inaccessible, which made analysis rather difficult. For instance, most of the students tended to use pronouns ('this one', 'that one over here', etc.) and from the tape-recording it was sometimes not possible to conclude what they were actually referring to. Video-recordings would probably have been able to provide some of that information.

All the teams began to solve the first task but they did not necessarily solve the rest of the tasks in the same order. The order was chosen by me. During two of the teams' attempts to solve the tasks it happened more than once that they asked if they could go on to one of the other tasks. In those occasions I did not allow them to proceed, but waited until their next solving strategy had failed, and then told them to go on to another task.

I chose not to use task based interviews, but only to observe the students while they tried to solve the tasks. The preliminary study showed me that interrupting the solution process – even if it was only to ask clarifying questions – was interpreted as if I indicated that they were using an erroneous solving approach or strategy. This was contaminating for the data.

### 3.2.5 Summing up

In the previous sections I have described the different parts in the empirical design, the intentions with each of them and how they contributed to the overall investigation. Because of the actual situation in the course the characterisation of solving strategies was reduced compared to my original intentions and focus shifted instead to the investigation of possible explanations for the students lack of abilities to solve the tasks.

The characterisation of the teaching practice was carried out as planned and I found that the observation matrix was a good tool for structuring my observations and for characterising classroom activity (that is, I found it fairly easy to categorise a certain situation into one of the categories).



### 3.3 Research design of the main study

The course investigated in the main study was an introductory mathematical analysis course. The subject matter was continuity, differentiability and integrability (Riemann integral) of one-variable functions. Twenty-four students were enrolled in the course. The exam consisted of a three days ‘take-home’ examination, where the students had to solve four tasks with subtasks again followed by an oral defence. This exam did not involve an external examiner, and the only grades used were pass/no pass. During fifteen weeks the class met twice a week for three hours.

#### 3.3.1 Classroom observations

Classroom observations followed the same method as I used in the pilot study, but based on experiences from the pilot study some categories in the observation template were changed and new ones included. The following categories were used:

Teaching/learning activities	Activities such as proof reviews, explanation of textbook content, and student group work
Task solving activities	Task solving in unison, in groups or alone
Extra instructional activities	Comments on for instance ways to prepare
Mathematical content	The mathematical content of the activity
Illustrations on the board	Illustrations presented on the board by the professor or the students
Solution strategies	Focus on strategies for proof production and other types of tasks
Techniques and schematic overviews	Overviews related to technical issues in proofs
Range of concepts, mutual relationships between concepts, perspective remarks	Elaborating on issues related to the development of students’ conceptual level
Reference to mathematics worked on in earlier courses or previously in the given course	Comments from the professor and students that link to old material
Anecdotes/detours, historical remarks	Comments and explanations going beyond the textbook

Navigation/description of agenda, motivation of results	The professors' explanation of the mathematical agenda
Enculturation	Enculturation in the mathematical society
Students' questions	Registration of students' question
Students' reactions	Registration of students' comments and questions
Professor's questions, comments and reactions	Questions posed by the professor, and his responds to the students

Beside some reorganisation of the content of a few categories, two new categories were included. 'Solution strategies' includes those cases where strategies to be used in tasks or in proof constructions are mentioned or discussed in class. An example could be (rhetorical) questions such as 'how should we approach a task like this?' raised by the professor and followed up by a discussion of a solution strategy. The second added category is 'Range of concepts, mutual relationships between concepts, perspective remarks' containing incidents where the class discusses or the professor mentions or talks about 'meta-mathematical' aspects such as how different concepts are related (for instance the relationship between norm spaces, metric spaces and topological spaces). The two new categories thus focus explicitly on the solving techniques, and on the promotion of conceptual understanding.

During the solving sessions, I attempted to observe one group of students solving a task at each lesson. The intention was to conduct non-participant observations so I did not interfere with the formation of the groups or the choice of tasks. The solution processes were either video or tape recorded. A lot of methodological problems occurred. Only about half of the students in the class agreed to be observed. This limited the selection to about five groups (the students often chose the same students to work with). Often there was much noise in the classroom, which made it difficult to record what the students were saying. It happened frequently that the students were actually not working together, but just sitting beside each other working on the same task or maybe on different tasks. Sometimes small talk led the students astray. These circumstances did not provide data material of a very good quality.

### 3.3.2 Preparation log

One of the main conclusions from the pilot study was that the professor intended to have a teaching practice based on dialogue, and the students wished that the teaching would focus less on the technical elements in the proofs and more on 'conceptual understanding'. Analysing the classroom observations it became clear that both the professor and especially the students actually felt a need to talk about the details in the proofs, and it was very difficult for the students to participate in the professor's attempts to discuss more general mathematical

issues.

To account for this discrepancy, I speculated that the students were unable to participate because they did not prepare for this kind of teaching method. In order to examine how students prepare for class I therefore initiated a survey of preparation habits in the main study.

Ten students volunteered to take part in this preparation survey. During the first couple of classes a preparation log was developed in co-operation with the students. An English translation of the log is listed in appendix H.

The record focused on three aspects. First the students had to record how much *time* they spent reading the textbook and solving tasks. Then they had to indicate *when* they read the text that the professor was going to address (before or after the professor's presentation). Finally, they had to specify *how* they read the textbook and *how* they worked with the tasks.

The students were committed to hand in a preparation form for each lesson during the duration of the course which amounted to twenty-three times (starting from the sixth lesson, with an omission of the 13th lesson due to 'self study', and ending at the second last lesson).

### 3.3.3 Semi-structured interviews

One set of semi-structured interviews was carried out during the observation period. The interviews were conducted over a one month period starting a month after the course had started. The students who were selected for interviewing were the ten students who had volunteered to fill out the preparation log for each class.

The students were asked about their expectations regarding learning mathematical analysis, how attending lectures and solving problems helped them learn mathematical analysis, and if they felt that it was necessary to come up with good ideas or tricks in order to solve the assigned tasks. In order to assess the reliability of the preparation log, the students were also asked how they experienced filling out the forms. An English translation of the questions is listed in appendix G.2. The individual interviews were audio-recorded and then transcribed.

Since the professor was the same as in the pilot study, I did not find it necessary to ask him all the same questions again. The interview with him thus concerned the changes that he intended to implement compared to the teaching conducted in the pilot study. The questions used in the professor interview are also listed in appendix G.2.

### 3.3.4 Response to interpretations of observations

As in the pilot study, the students and the professor were presented with my interpretations of the classroom observations. The evaluation and the response to my interpretations were also used as data to characterise the teaching practice, and the students' views on it.

### 3.3.5 Constructed task solving sessions

Eight of the ten students who filled out the preparation log volunteered to participate in constructed task solving sessions. The solving sessions were placed after the last class, but before the final written exam. This time, the students were asked to solve only one task. This choice was made based on experiences from the pilot study where a majority of the students had spend two to three hours working on the tasks without reaching acceptable solutions. I designed a task that in my view was easier than the tasks used in the pilot study, an English translation of the task is listed in appendix D.3. The task in the main study was a proof task. It was a variant of a textbook task that the professor had asked the students to solve, so the task resembled the tasks that the students were used to solve. The task had the advantage of including many of the central concepts and the use of illustrations would be a powerful tool in reaching a solution. Furthermore, a solution to the task demanded that the students showed overview of the concepts involved and their relations. The task could not be solved merely by implementing a known procedure or algorithm, or by copying an example (if it, contrary to expectation, turned out that some of the students had in fact solved the textbook variant it would not be possible to copy that solution). Finally, the solution involved a combination of several definitions and theorems from different parts of the textbook.

The students were told in advance that they could regard the task as preparation for the written exam. In that way they knew the content of the task was within the content of the course, and that the task in style would resemble the exam tasks. As in the pilot study, I decided to pair the students, and in all four groups students who normally worked together were paired. As a result of this division strategy the students were by and large on the same level. Based on experiences from the pilot study with this particular research activity, I chose to video-record the four solving sessions, instead of making only sound recordings. During the observations I took notes for the subsequent data analysis.

### 3.3.6 Summing up

In the previous sections I have described the different parts in the empirical design used in the main study, the reasons for including the respective research activities, and the rationale behind the design of these activities. Some of the research activities from the pilot study have, with some modifications, been used in the main study, some activities have been toned down (for instance the number of interviews), and some additional investigative activities have been included (survey of preparation habits).

These descriptions will be used in the next sections to discuss ways of combining empirical data, and how the notions of validity, reliability, and generalisability can be accounted for within the research design presented.

### 3.4 Combining data generated from different methods

The research design is characterised by being a multi-methods design [Cohen & Manion, 1994, p. 233]. In order to answer the formulated research question different qualitative and quantitative methods have been used. As described above, the research includes classroom observations, task solving observations, semi-structured interviews, constructed task solving observations, and preparation logs. By some, the multi-methods approach is called triangulation, *methodological triangulation*, because the same phenomenon is being ‘attacked’ from different angles [Cohen & Manion, 1994, p. 233-235]. Methodological triangulation is a way to capture a complex phenomenon, but triangulation is in the literature also used as a way to validate data. For instance, interviews with research objects can be used to verify if the researcher’s interpretations of observations are correct [Bryman, 2001, p. 275]. This is also known as respondent validation [Bloor, 1978, p. 548-550]. I attempted to use a sort of collective respondent validation and presented my interpretations of the classroom observations to the students and the professor in the pilot study and in the main study. Assuming that the research subjects feel free to express own opinions and are capable of criticising the researcher’s interpretations the responses can be used to verify the analysis *if* the research subjects agree. If they disagree, the researcher can use their critique to look at the analysis one more time and possibly make adjustments, but the responses can – in my view – not be used to dismiss the analysis all together.

### 3.5 Validity, reliability, and generalisability

The notions of validity, reliability, and generalisability represent three aspects of the assessment process of a research project’s research design, methods, data analysis and conclusions [Mason, 2002, p. 38-39]. These notions come originally from quantitative research paradigms such as the natural science paradigm(s), but they have been very influential in qualitative research paradigms, although qualitative researchers still discuss the usefulness of the notions for assessing the quality of qualitative research [Bryman, 2001, p. 272]. All the three notions can be applied to the methodology as well as the findings. Validity, reliability and generalisability of the methodology can be accounted for before introducing the research results, whereas it is more natural to discuss their counterparts related to the data analyses and the findings after the presentation of the results. The following questions characterise the two parts of the three notions:

- **Validity of methodology** If carried out in the best way possible, can the methods used provide an answer to the research question?
- **Validity of findings** Do the findings provide a valid answer to the questions posed?
- **Reliability of methodology** How well do the methods and tools used in the investigation work?

- **Reliability of findings** Are the findings reproducible?
- **Generalisability of methodology** Do the methods used allow generalisability?
- **Generalisability of findings** Can the analysis, results, and explanations be generalised?

Validity (methodology) relates the research questions to the research design. When research questions have been formulated how will the researcher go about examining the questions and are the chosen methods relevant in the pursuit of answers? Can the given methods provide answers to the research questions? In the introduction and in the previous sections in this chapter I have tried to explain and justify the design of the different parts of the investigations, the rationale behind the choices and how the specific parts relate to the research questions. Through this description, the validity of the investigation has been accounted for.

Validity (findings) concerns to what extent a research finding is what it claims to be. A study can have been designed and carried out in a way which ensures methodological validity and reliability, and still the results of the study might not provide a valid answer to the posed research question. An example could be that a student filling out the preparation logs misinterpreted the categories. Then the result would not provide a valid answer, even though measures had been taken to ensure that the student would be able to distinguish between them (providing high methodological validity).

Reliability (methodology) is more problematic to use in qualitative studies since in its original sense it asks for some sort of measurements of how well measuring tools are working. This implies that qualitative “*methods of data generation can be conceptualized as tools, and can be standardized, neutral, and non-biased*” [Mason, 2002, p. 187], which is not the case. Instead the discussion of data reliability should concern the researcher’s abilities to carry out the research study in a “*thorough, careful, honest and accurate*” way [Mason, 2002, p. 188]. Bassey [1999] and Schoenfeld [2007] agree with this, and suggest instead to introduce the notion of *trustworthiness* of the research process [Bassey, 1999, p. 74-75][Schoenfeld, 2007, p. 81-88]. To demonstrate that the research study is reliable (or trustworthy) the researcher must be able to convince others that the study has been carried out in that manner. This is done through a thorough description of the research process. With the risk of sticking my head into ‘the lion’s mouth’, I see a similarity between a similar request in the publication of experimental natural science studies. In such studies researchers are obliged to report the experimental setting in such a way that other researchers (in principle) are able to replicate the experiment. Within educational research it is not possible to demand the same degree of replicability since the research objects and the settings can never be repeated [Bryman, 2001, p. 273], but the demand for transparency in the research methods and the performance of the research can be transferred to the assessment of qualitative research studies. This is the mean-

ing I ascribe to the request for methodological reliability. As mentioned in the introduction, writing a monograph which provides enough space for a thorough description of the methodology is an attempt to ensure methodological reliability of the investigation.

Reliability (findings) concerns to what extent the findings presented are reproducible. Educational research involves the study of people, and the findings are to a high degree produced through analyses and interpretations. If the results of a study are to have a high level of reliability, it means that it (in principle) will be possible to reproduce the findings, because the analyses and interpretations are reliable and lead to the conclusions in a natural way. One way to enhance the reliability is to have several researchers analyse and interpret the data, independently. If they reach the same results, the findings would have a high degree of reliability. Another way to secure high reliability of the findings is to make the presentation of the analyses, interpretations and conclusions as clear and detailed as possible. This makes it possible for others to check the steps leading to the conclusions.

The notion of generalisability in its originally form is also very difficult to accomplish in qualitative research studies based on case studies. If one should be able to claim generalisability in a case study it would be necessary to find a way to select typical cases as representatives for larger sets of objects. As Bryman states, this is not possible:

It is important to appreciate that case study researchers do not delude themselves that it is possible to identify typical cases that can be used to represent a certain class of objects. (...) In other words, they do *not* think that a case study is a sample of one. [Bryman, 2001, p. 51]

The cases used in the current study (pilot study and main study) have been chosen because they made investigation possible! The preliminary study gave insight into the different ways real analysis courses are taught at different institutions. Based on these experiences a specific course was chosen. The course was structured in such a way that dialogue between students and professor was possible (not just dialogue between students and teaching assistants), and it was chosen because of the emphasis put on the notion of proofs. So in this way, the case is what Bryman refers to as an “*exemplifying case*” [Bryman, 2001, p. 51], which is a case that allows the researcher to make investigations that can be used to answer the posed research questions.

Based on my experience (observation of three Danish courses and one Canadian) the chosen course was *not* typical. In fact, it would not have been possible to select a typical case based on what I have observed. In pairs, the four courses were alike in structure (lectures held by a professor and tutors in charge of solving sessions or lectures and solving sessions held by the professor). In two of the courses about sixty students attended the lectures, whereas this was not the case for the two remaining courses. Three of the courses put great emphasis on

proofs, but the diversity in the professors' perspectives on proofs and the learning of proofs were different. In two of the courses the impression was that the students found it valuable to attend lectures whereas this was not the case for the two other courses. I speculate that this difference between courses would still exist even if I had had a dozen more courses to choose from. What is typical or general about the chosen case is that it is an analysis course offered at a university.

Generalisability (findings) concerns to what extent the conclusions drawn on the basis of the case are general. If the conclusions are based on or concern aspects which are very specific to the chosen case, the findings might lose their generalisability. An example could be the finding that the students' experienced difficulties solving mathematical tasks because they used all of their preparation time on (unrelated) project work.<sup>3</sup> Such a finding is very connected to a specific characteristic of the case investigated, and thus not a general result (although the finding that lack of preparation leads to solving difficulties is general in nature).

### 3.6 Summary

This chapter presented and discussed the different parts of the empirical research design. Focus has been on why I chose to investigate the research questions the way that I did, how the different parts in the research design contribute, and why other ways of pursuing the research questions have not been followed.

The investigation is primarily empirical, and includes methods such as classroom observations, individual interviews, task solving observations, and study habit logs. Analytical work has been carried out in the development of the observation template and the research hypothesis.

Doing research will always have limitations, some of which have already been discussed, but not with reference to the actual conclusions of this study. Later, in the discussion chapter, I return to the methodology and discuss how the research design has affected the conclusions, which limitations the design has induced, and what actions could have been taken to avoid this.

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<sup>3</sup> Project work is mandatory at the university where the pilot and the main study were carried out, and shall occupy 50 % of the students' study time.





## 4 Developing a hypothesis

“...the task of learning and teaching mathematical justification conflicts with the pursuit of learning and teaching mathematical relationships, concepts and procedures in a flexible manner.” (Tommy Dreyfus)

The pilot study focused on identifying essential features and elements in the teaching practice as well as in the solution processes which could provide good characterisations of both aspects. The idea was to compare these characterisations in the hope of revealing how the teaching practice influences students’ solving processes. From the beginning, the pilot study was designed to examine the research question originally intended. But since the students in the course found it very difficult to solve the assigned tasks, the study ended up focusing on the main research questions.

The outcome of the pilot study was partly the development of a tool for characterising teaching practices and partly the development of a hypothesis concerning if and in what ways the students’ difficulties with task solving were caused or influenced by the teaching practice. This chapter describes the second outcome, i.e. how the hypothesis emerged from the data analysis.

This chapter begins with a presentation of the professor’s and the students’ views on teaching and task solving, and their opinions about the actual teaching practice. A short excerpt from the classroom observations serves as an illustration of the kind of dialogue that takes place in the classroom.

Next, the students’ solution processes are analysed with the aim of locating the main reasons for their difficulties with task solving, thereby examining the first research question. The results from this analysis combined with an analysis of the problems related to the teaching practice leads to the development of a theoretical framework useful for analysing proof validation situations and proof construction situations. Using this framework as its basis, a hypothesis is formulated at the end of the chapter. The hypothesis focuses on problems related to the development and creation of a mathematical overview and the understanding of mathematical details.

## 4.1 Task solving difficulties and the teaching practice

A month after the course (in the pilot study) started, the professor takes the initiative to talk to the students about their task solving performance. He is under the impression that the students do not manage to solve enough tasks, and he is interested in knowing the reason in hopes of improving the situation. He also wants to send the message that he expects them to solve at least four tasks from each chapter.

The students' immediate response is that solving the tasks is time consuming, and they do not have the needed time available to solve all the assigned tasks.

### 4.1.1 Students' views

This collective response is in line with what the individual students reply in the interviews when asked if they felt they solved too few tasks. Most of the students interpret 'too few' from their level of ambition and the time they have available to spend on preparation. One student says that right from the beginning of the course she gave up trying to solve all the tasks the professor had assigned. She does not have the time, and indicates that she does not have the ambition either. The latter implies that she believes that since she does not strive to achieve top marks, it is 'okay' not to solve all the assigned tasks. Other students also concur with the professor that they do not manage to solve enough tasks, but this does not seem to frustrate them. It sounds as if they accept the situation as it is because it is not in their power to change it. A third group of students interpret the professor's message as a disciplinary statement designed to push them to spend more time on task solving. A student expresses this view by saying that he feels that he spends enough time on task solving so he did not pay much attention to what the professor said.

About two-thirds of the students state that they find the tasks difficult. Some say they feel they need to use tricks and find good ideas to solve them; they try to use the textbook to provide the ideas. Some students explain that when they have to give up solving a task it is because they have no idea whatsoever about how to approach it, while others find it difficult to combine information provided in the task. Thus, even though the students generally do not find the tasks *too*<sup>1</sup> difficult, most of them experience difficulties solving the tasks, and based on their own interpretation of their study efforts, it is not because they do not spend *enough*<sup>2</sup> time.

Many of the students view task solving as an important part of learning a mathematical subject, but the reasons for this opinion differ. The answers include

<sup>1</sup> In the sense that they are frustrated and find the situation unacceptable.

<sup>2</sup> The students seem to define what is *enough* in relation to how much time they have available and not according to how much time they need to spend to reach a desired level. This is why, in the main study, I found it necessary to examine how much time students actually spend on course preparation.

purposes such as: gaining confidence about the mathematical content, a way to remember mathematics, learning to use mathematical tools, understanding why different proof strategies are used to prove different statements, gaining proving experience, practising locating results in the textbook, knowing how theorems are related to previously acquired knowledge, constructing mathematics actively promotes learning, gaining practical understanding, learning to formulate mathematics rigorously, practice for passing exams, controlling one's understanding of the text, realising the mathematical structure composed of definitions and theorems.

A majority of the students experience a gap between the lectures and their abilities to solve the assigned tasks. This opinion is shared by both those students who experience difficulties with task solving and those students who generally feel capable of solving the assigned tasks. The professor's presentation of the subject matter is generally viewed to be very focused on technicalities in the proofs. One student provides the following description:

I often find that in the presentation of the chapter, the professor ends up paying too much attention to technical matters, at the expense of the conceptual stuff... the focus is so much on the technicalities, which I would be able to figure out by myself, but the conceptual things, which is what I need, are drowned out by the technical issues. (Chris, student)

A few students experience the lectures as a help when solving tasks, while a large group of students feel that their solving difficulties could be eased if the professor also paid attention to issues such as why a certain proof strategy is used to prove a theorem, and why the proof actually proves the theorem. In order for the lectures to be a direct means for task solving, some students say that the professor should present more examples and solutions to tasks. Other students believe that the teaching approach should focus on promoting "*conceptual understanding*" and "*overview*" instead of paying so much attention to proof details, and that this would help them in task solving situations.

#### 4.1.2 Professor's views

Not many professors who teach mathematical analysis with a focus on proofs would be likely to argue that it is a bad idea for students to develop conceptual understanding and overview, and knowing why certain proof strategies are appropriate in proving certain statements and why the proofs actually prove the statements. As I see it, this particular professor is no exception, although some of his opinions expressed in the interviews are contradictory. The following will clarify how.

In general, the professor aims for a teaching style that allows room for discussions about the mathematical content. This means that the students must prepare ahead of time to the point where they should be able to:

... pose questions and be exposed to the concepts a bit. How far do they reach to find out what this concept means. Partly, what is the range, the limitations, what does it do, why don't we do it like this, wouldn't it be more logical or what are the pitfalls? (Professor, pilot study)

This clearly puts an emphasis on the mathematical concepts. When asked about why he reviews proofs in detail, he begins by explaining that he wants to show the students that mathematics is not a collection of facts that they simply have to memorise. They should experience that it is possible to deduce all the results "*basically from scratch*". This points to an emphasis on the mathematical structure of axioms, definitions and theorems, and plays down how to apply the individual statements.

Later in the interviews, the conceptual and structural focus are relegated into the background when he promotes a view of mathematics as a toolbox. He talks about proof strategies (such as proof by induction) and specific argumentation structures (such as Cantor's second diagonal argument) as examples of tools. The purpose of the course is to introduce the different tools in the tool box, show how they work and in what situations they are applicable. He does not expect the students to be able to use 'a screwdriver' by the end of the course, but it is important for him to show them that it exists so that later in their mathematical careers when they need to 'tighten a screw', they do not have to invent the tools to do it. In the professor's view, reviewing proofs also provides the students with a 'foundry ladle' they can use when they construct new proofs.

Before a lesson, the professor prepares in such a way that he is able to explain the details in all the proofs in the chapter in case the students ask questions for clarification. He basically wants to let the questions from the students determine what he talks about. He is not clear about where and how the (conceptual) dialogue, which he also wants to conduct, fits in. He has not considered whether such a dialogue is even possible given the students' preparation, nor whether he expects himself or the students (through their questions) to initiate it. At the course evaluation at the end of the course, he expresses dissatisfaction with the kind of dialogue that took place in the classroom and also with the students' level of participation. Thus, the resulting teaching practice did not quite correspond with his objectives.

#### 4.1.3 Example of classroom dialogue

From the previous description it is possible to conclude that one group of students wants more focus on the development of conceptual understanding, and less focus on the clarification of technical details. The professor would also like to discuss concepts and to include the students in concept related dialogues. But he also believes that it is important that the students experience how the mathematical structure of analysis can be constructed by means of proof. The question is then whether the latter goal dominates the teaching practice and causes the relative

neglect of conceptual issues.

The aim of the following example and subsequent analysis of a classroom dialogue is partly to show a typical dialogue and partly to demonstrate that student behaviour is actually a factor when it comes to focusing on the technicalities of the content matter. The following excerpt is taken from the beginning of a class where the main topics were derivatives, differentials and tangent planes. A student asks if an equality sign appearing in the proof of Theorem 11.20 (see appendix C.2) is wrong. The equality sign is in the expression  $|T_2(\mathbf{h})| = \|Df(\mathbf{a})(\mathbf{h})\| \|g(\mathbf{a} + \mathbf{h}) - g(\mathbf{a})\|$  ( $T_2$  is defined as:  $T_2(\mathbf{h}) = (Df(\mathbf{a})(\mathbf{h})) \cdot (g(\mathbf{a} + \mathbf{h}) - g(\mathbf{a}))$ , where  $\mathbf{a}, \mathbf{h} \in \mathbb{R}^n$ ,  $f$  and  $g$  are vector functions, and  $D$  is the total derivative):

- 2     **Professor:** So we are going to to talk a little bit about differentiability  
and differentials today. (He browses through the textbook.) Theo-  
rem 11.20, are there any comments? (The professor ties his shoelaces  
4     and starts to erase the blackboard.)
- 6     **Betty:** Isn't there missing a, eh, an inequality sign in 11.20? (They discuss  
where in the proof she thinks the sign is missing.)
- 8     **Professor:** That's a good question. (The professor keeps on erasing the  
blackboard) Can we get something on the table, why must there,  
mustn't there? You say, Betty, that there has to be, because?
- 10    **Betty:** Because when you look at  $T_2$  in the middle of page 340, then it's  
defined as the total derivative of, of ...
- 12    **Professor:** Of  $f(a)$  applied on  $h$ .
- 14    **Betty:** Yes, yes exactly. Multiplied with this other expression. And then  
you can use this Cauchy-Schwartz and ... split them, right? And then  
I think it has to be there.
- 16    **Professor:** So it's in Cauchy-Schwarz inequality. What, what does it say?
- 18    **Alan:** It says, it says something.
- 20    **Betty:** If you have a product of something.
- 22    **Professor:** If you have a product.
- 24    **Betty:** Yes. Then it's smaller than the product of the norms of each one.  
Smaller than or equal to.
- 26    **Professor:** Yes. So if it's Cauchy-Schwarz then it has to be smaller than or  
equal to, right.
- 28    **Betty:** Yes.  
(after a short break)
- 30    **Professor:** Has something surprised you about this theorem? (Nobody an-  
swers) No? The first two items, you could appropriately call them the  
linearity of the total derivative, right? What would you call the last  
one? (The professor is still erasing the blackboard) Do you have a  
name for it?
- 32    **John:** The product rule.
- Professor:** The product rule, it's also called the Leibniz rule. So should we  
leave 11.20 alone? (No one reacts)

The professor starts with an invitation to the students, “*theorem 11.20, are there any comments?*”. It's not easy to deduce what his intentions with the question are. A student perceives it as an invitation to pose a question concerning a technical matter in the proof of the theorem. The professor's response indicates

that the student's reactions to his question are acceptable, establishing the sociomathematical norm that an open question from the professor can be responded to with a question concerning a detail in the proof (I will return to the establishment of social and sociomathematical norms in chapter 5).

The professor encourages the student to answer her own question and his role is to provide authority. After solving the dispute about the inequality sign, the professor continues with the same type of question (line 26) as the first one, but now he has to reveal what insight he was aiming at because none of the students respond (lines 26-30). It turns out that he wants to direct the students' attention towards the linear property of the total derivative and that the total derivative satisfies the product rule. The realisation that the three properties in the theorem can be interpreted in this way could strengthen the students' statement image of this particular result and also advance their concept image of the total derivative. Furthermore, it connects the total derivative to the more familiar notion of one variable derivatives.

In conclusion, the professor's questions to the students are designed to develop statement images and concept images and establish a connection between a new, unknown statement and an already known result. He is thus aiming to promote conceptual understanding. The students, on the other hand, focus on proof technicalities and are only able to answer very specific questions (such as "*Do you have a name for it?*", line 29).

#### 4.1.4 Students' views on classroom dialogues

Most of the students (two exceptions) indicate that they find it difficult to answer the professor's questions. They have trouble finding out what kind of answer he is looking for and most of them feel uncomfortable saying something possibly erroneously in front of the whole class.

One of the students who wants more of a focus on conceptual matters explains during the interview that he does not want this to happen through dialogue. It is the professor alone who should focus more on concepts than on proof details in his lectures. Other students agree with this position based on the assumption that it is impossible to have a real discussion when they are having trouble understanding the textbook. In addition, even though the students agree that asking clarifying questions concerning issues in the text is permissible, it can be hard to pinpoint the difficult spots:

I haven't really known what to ask when I don't understand anything. Saying I don't understand the whole theorem, I think that's like, then it can go really slow, if you have to go through it all over again. And I often feel that I understand what is going on when Michael constructs a proof for a theorem, but maybe I haven't really understood it when I get home. (Bill, student)

This student feels that it is difficult to formulate specific questions when he

reads the text. Even though he has a sense of understanding when the professor reviews a proof in class, he is not at a level where he can formulate additional questions which go beyond the details. This prevents him from taking active part in the lectures.

The student interviews indicate that it is difficult to carry out dialogues in class if they have not understood the details in the text. At the same time, however, trying to understand the proof details when the professor goes through the proof is also hard and many of the students often experience that they have not gained a sense of understanding of the content even after the professor has explained the proofs.

#### 4.1.5 Summary

It became clear that the students do not manage to solve the expected amount of assigned tasks during the amount of time they feel they have available. This implies that they might have unrealistic expectations about how much time they need to spend and/or that solving the individual tasks takes too much time, because they are too difficult. If the former reason for not solving enough tasks is the cause, then there is one straightforward way of resolving their solving difficulties: spending more time on the tasks. If the latter reason is the cause, it would be interesting to examine what mathematical difficulties the students encounter during a solution process which prevents them from reaching a solution or which drags out the process. This examination is carried out in the next section.

A contradiction is detected between what the students feel they need in order to understand the mathematics and to be able to solve the tasks, on the one hand, and how they act in the classroom, on the other. My hypothesis is – and this will be elaborated upon further following the analysis of the solution processes – that this contradictory behaviour is a result of the dichotomy between focusing on ‘the bigger picture’ and examining ‘the individual details’. Based on a comparison of the views expressed by the students and their behaviour in class, it could be argued that it is difficult to have a focus on and an understanding of the mathematical structure, the concepts and the relations between them when the proof details are not understood; likewise, a focus on the comprehension of the proof details without knowing, for instance, what the theorem says, and how it fits in with the other theorems, makes it very difficult to understand the proof.

## 4.2 Solution processes

In total, the pilot study contains sixteen episodes of students solving tasks, four teams solving four tasks (the tasks are listed in appendix D.2), which amounts to more than fifty pages of transcription. Although content analysis of all sixteen processes has been undertaken, it would be quite an ordeal for the reader if the analysis of all the processes were to be presented here in detail. Consequently,



I have chosen to focus mainly on the first task, where I present the full solution processes of teams B, C (partially) and D, as well as the first half of the solution process of team A.<sup>3</sup> First, the task is presented and then analysed, followed by a description and interpretation of the processes. Next, I compare and contrast the four processes, at which point I also draw on analyses of the other solution processes, although the empirical foundation of these analyses will not be presented to the reader. In order to compare the solution processes related to task 1, I also present parts of the solution process of team C when working on task 2.

The pairs of students in three of the four teams were, as mentioned in the methodology chapter, at the same level, whereas the two students in team D were on very different levels. Two years earlier, the students in team C had taken the preceding analysis course, though taught by another professor. During the student interviews, all of the students (except Bill in team B) describe themselves as active students who spend a considerable amount of time on course preparation.

#### 4.2.1 Task 1

The reader is encouraged to try to solve the task before proceeding to the analysis of the solving processes. This will familiarise the reader with the task and also put the reader in a better position to understand the different stages the students go through trying to solve it.

##### Task 1

Let  $(M, \sigma_{discrete})$  and  $(M, \sigma_d)$  be two metric spaces, where  $\sigma_{discrete}$  and  $\sigma_d$  are the discrete metric and an arbitrary metric, respectively. Let  $i$  be the identity, i.e.  $i(x) = x$ ,  $x \in M$ .

Determine if the mapping  $i : (M, \sigma_{discrete}) \rightarrow (M, \sigma_d)$  is continuous and uniformly continuous.

Before presenting this task to the students, I had no inkling that they would find it difficult to choose a strategy (e.g. choosing to use the epsilon-delta definition of continuity), but I suspected that carrying out the strategy might give rise to difficulties. In the formulation of the task, three different functions are explicitly mentioned, namely the two metrics and the identity function, and one set  $M$  with no further specifications. By using the symbols  $i : (M, \sigma_{discrete}) \rightarrow (M, \sigma_d)$ , it is explicitly stated that the discrete metric space is the domain of the identity function and that the metric space with an arbitrary metric is the codomain. The set theoretical domain of the two metrics is the same, namely  $M$ , which

<sup>3</sup> Because Team A's attempt to solve task 1 is seven pages long, it is not possible to present the entire process here. The excerpt selected, I believe, satisfactorily shows the characteristics of the process and the students' difficulties.

indirectly follows from the notation  $(M, \sigma_{discrete})$  and  $(M, \sigma_d)$ , but the ranges of the metrics are not stated in the task and one of them, the range of an arbitrary metric, cannot be determined since the metric is not specified.

The solution of the first part of the task (the question about continuity) can follow one of two strategies: either the epsilon-delta definition of continuity can be used or the formulation of continuity in terms of open sets. The second part of the task (concerning uniform continuity) can only be solved using epsilon-delta arguments since uniform continuity is not a purely topological property.

#### 4.2.2 Team D

First, the solution process of team D, which consists of Danny and Dylan, is presented since it compactly reveals how the task can be solved. Figure 4.1 contains Danny's notes. After spending 20 seconds to read the task, they begin talking:

The image shows handwritten notes on a piece of paper. The first line defines the discrete metric:  $\sigma_{discrete}(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ . The second line states the definition of continuity: "Kont. i.e.  $\Rightarrow \forall \epsilon > 0 \exists \delta > 0 \forall x, y \sigma_{discrete}(x, y) < \delta \Rightarrow \sigma_d(x, y) < \epsilon$ ". The third line starts the definition of uniform continuity: " $\forall \epsilon > 0 \forall x, y \in M \exists \delta > 0 :$ ". The fourth line shows the implication for uniform continuity: " $\sigma_{discrete}(x, y) < \delta \Rightarrow \sigma_d(x, y) < \epsilon$ ".

Figure 4.1 Danny's notes from task 1.

- Danny:** Mmm, do you understand the task, Dylan?
- 2 **Dylan:** Mmm, yeah, I think I just want to check the definition for the discrete, I think.
- 4 **Danny:** Okay, I remember that one. This  $\sigma$ , the one you have called discrete. (He writes down the definition) Eh,  $x \dots y \dots$  (He shows it to Dylan)
- 6 **Dylan:** Okay.
- 8 **Danny:** Do you? This is the clue, I can tell, it's that continuity both depends on the metric in the space you come from and in the space you are going to. The mapping, eh. Then one can imagine that strange things happen even though the mapping does nothing when you change metric. Now I'll write down the definition of continuity. (He writes it down) Okay.
- 10
- 12

- 14 **Dylan:** But this is the definition of uniform continuity, you have written  
down.
- Danny:** No, it shouldn't be. I can check it.
- 16 **Dylan:** The way you have written it, you have delta independent of your  
 $x$ . If it's continuous, it's also dependent on  $x$ , uniform is independent.
- 18 It's not so important.
- Danny:** Yes it is, because we are being asked about the difference between  
20 continuity and uniform continuity. So let's get it right (He looks in the  
textbook).
- 22 **Danny:** You mean that 'for all' are going on the other side here?
- Dylan:** Yes, this is uniform.
- 24 **Danny:** Okay. Fine. Now, it's a bit annoying that the function goes the way  
it does, because to say that sigma-discrete is less than epsilon, that is  
26 easy. Eh, but it's over here, here we know a whole lot because ... it  
behaves so simple, the discrete metric.
- 28 **Dylan:** Yes.
- Danny:** (He whispers) No, and this has to be  $f, y$ , but that is almost, that  
30 is the same.
- Dylan:** Mmm.
- 32 **Danny:** Well, this one you don't have to write. So if ... Okay. I have it.  
(He laughs) If we take an arbitrary epsilon and set delta equal to  $\frac{1}{2}$ .
- 34 Then this, then this can be fulfilled only if  $x = y$ . Then this (meaning  
 $\sigma_d(x, y) < \epsilon$ ) is automatically fulfilled and therefore it's continuous.
- 36 **Dylan:** Yes.
- Danny:** And ... furthermore, you can see that it doesn't make a difference  
38 where you put this quantifier so it's also uniformly continuous.
- Dylan:** Mmm. Yes.

The process takes about eight minutes. First, Danny quickly realises the point of the task, namely that the property of continuity depends on the structure of both the domain and the codomain. He has no trouble choosing a solution strategy, and without any apparent motivation or consideration, he starts to write down from memory the definition of continuity in a metric space. Then an exchange follows. While Dylan is not mathematically strong (based on observations from both the preliminary and the pilot study), he nevertheless spots that Danny writes down the definition of uniform continuity instead. Danny is not immediately convinced that he made a mistake and Dylan tries to diminish his criticism by saying that "*it's not so important*". But Danny is able to judge that it is important to separate between the definitions of continuity and uniform continuity. He corrects the definition after having checked the textbook. Then a period of strategy implementation follows where Danny tries to apply the definition, and suddenly he sees that choosing  $\delta = 1/2$  forces  $x$  to be equal to  $y$ , which makes the difference between the images zero (the images are  $x$  and  $y$ , respectively, since the function is the identity). He correctly spots that the position of the quantifier,  $\forall x$ , (he writes  $\forall x, y$  instead – which presumably is a 'leftover' from his first version of the definition of continuity) does not influence the line of argumentation, but when writing down the argumentation, he places the quantifier in the wrong position.

The solution process proceeds without any difficulty. It is only possible to hypothesise about the influence the identification of the main point has for the solution process. Although the definition of continuity operates with two different metrics, the students have never been presented with tasks where they explicitly have to make use of two different metrics. Thus, the task is not an exercise or a routine task, but the fact that Danny quickly identifies the main point seems to turn it into a routine task for him. After having identified the main point, it is only a question of checking the definition. Dylan's behaviour is an example of person-guided reasoning (although he does not receive many opportunities to reason, he is able to notice incorrect reasoning which means that he is able to follow Danny's line of argumentation).

### 4.2.3 Team C

Team C, made up of Chris and Curt, also obtains an answer to the task, and identifies the main point, although using a different formulation. Figure 4.2 contains Chris' notes. After having read the task for about three minutes, they immediately discuss a solution strategy:

- 2       **Curt:** Thought that, should one assume that ... if one should show that  
               this is continuous, if the inverse image of an open set is open.
- 4       **Chris:** Yes, that was what I was thinking about. It must have something to  
               do with open sets.
- 6       **Curt:** Yes, so one could imagine that if we had an open set in one of the  
               sets [yes] and then try to imagine what the inverse image would look  
               like.
- 8       (pause)
- 10       **Chris:** When it says  $M$ , is that then the same  $M$ ?
- 10       **Curt:** Yes.
- 12       **Chris:** It maps the space onto itself with a different metric.
- 12       **Curt:** Yes.  
               (pause, they write)
- 14       **Curt:** When it's the identical mapping you stay in the same set, but you  
               change metric. But it is the same elements [yes]. So for every  $x$  the  
 16       image is the same, that is the inverse image for every element.
- 18       **Chris:** And the question is if openness, that is we have an open set, the  
               question is if openness changes [yes] when we change metric [yes]. The  
 20       discrete ... (He looks in the textbook). This is the arbitrary metric,  
               this could be the discrete metric, for instance, right? [It could be] It  
               was just to look at a special case for a start, it might be enlightening.
- 22       (pause)

Both Curt and Chris associate the formulation of the task with the topological definition of continuity. After having 'decided' what strategy to use, they individually clarify the set-up (Chris in line 9 and line 11, Curt in lines 14-16), and this leads to Chris' formulation of the main point (lines 17-18). Hereafter the students try to determine what open sets look like in the two spaces. Chris starts to look in the textbook because he recalls a property of the discrete space:

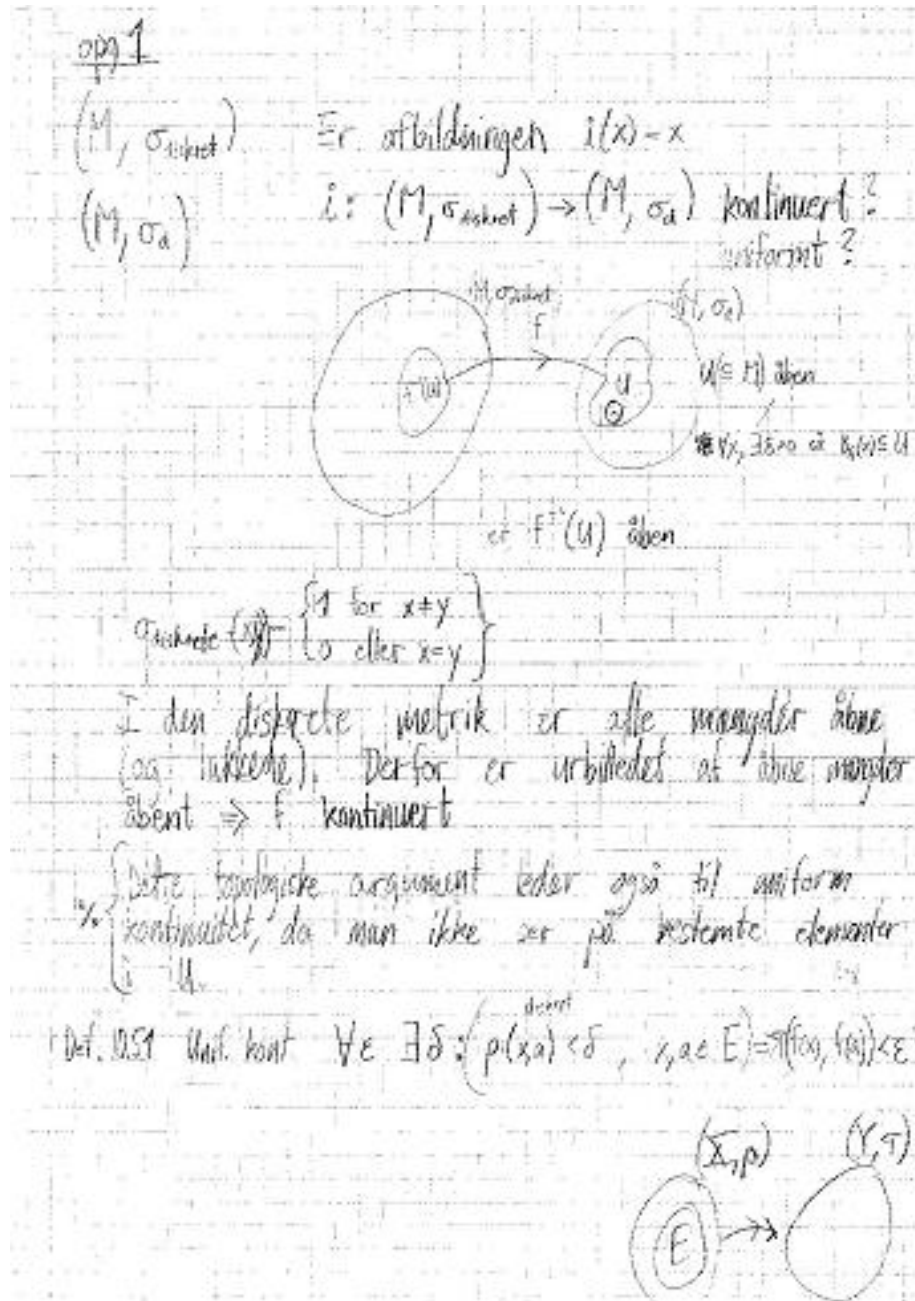


Figure 4.2 Chris' notes from task 1.

- 42       **Chris:** This other day, when we, we have only repeated measure and in-  
 44       tegration theory so far, there we also had something with open and  
       closed sets. I remember something, there was an example here, saying  
 46       that in the discrete, in the discrete space all sets are both open and  
       closed. That's what I remember and now I have looked it up. I just  
 48       have to check if the discrete space is, it's I guess, a space with the  
       discrete metric, I guess it is. (He looks in the textbook) Yes in the  
       discrete metric all sets are both open and closed [yes]. But then the  
       task is basically solved.
- 50       **Curt:** But you don't even have to assume that it's the discrete metric over  
       here (in the codomain) [No, no] because you just say that if you have  
 52       an open set over here ..
- 54       **Chris:** Yes, no matter if it's open or closed the inverse image will be open  
       because they are both open and closed. So at least, the answer is that  
       it's continuous. (They write down) Yes, and that is because in the  
 56       discrete metric the ball is equal to the point [yes] and that is of course  
       in the set.
- 58       **Curt:** Yes, or you could say that you can put a ball, you just have to have  
       radius a half, right? [Yes] and put it on whatever [yes] and the union  
 60       of open sets is open.

Chris' first comment provides an explanation for why both students favour the topological definition: they have at this point only revised measure and integration theory for the exam. This topic was introduced in the professor's notes, which define continuity in a topological space. This means continuity is defined in terms of open sets and not distances/metrics.<sup>4</sup> Chris remembers the set property of the discrete space that all sets are both open and closed, and realises immediately that the first part of the task is solved. Curt needs to check that the solution does not depend on the special case they considered (where both spaces are assumed to be the discrete space), and Chris confirms this. After writing down the solution, it seems that they both need to justify their solution (lines 55-60). What Chris says in lines 55-57 is that an open ball around a point in the domain (with a radius less than one) consists of only this point and that it is contained within the open ball (with radius epsilon) in the codomain (he calls the open ball in the codomain for "*the set*", line 57). This is an informal version of the epsilon-delta definition of continuity. Curt agrees but returns to the topological proof with an argument for why the inverse image of an open set is in fact open in the discrete space (lines 58-60). The first part of their discussion takes eleven minutes. Even though Chris has already sketched the epsilon-delta proof, the two students struggle to solve the second part of the task. Chris and Curt start discussing whether uniform continuity can be formulated in terms of open sets:

- 80       **Curt:** But you have to have a distance, right?  
       **Chris:** Now I will just look in the textbook.

<sup>4</sup> In a metric space the formulation of continuity in terms of open sets becomes a theorem.

- 82 **STO:** To be inspired?
- 84 **Chris:** Yes, but I will look at the definition of uniform continuity, for inspiration. (He looks in the textbook). If delta is chosen to be  $1/2$  then it also satisfies that [yes] all deltas are smaller than one, it also satisfies the conditions for uniform continuity. (He keeps on browsing through the textbook) I could have written this solution a bit clearer.
- 86 **STO:** So you agreed with yourself that it was ...?
- 88 **Chris:** Uniformly continuous. This argumentation is independent of which  $x$  we look at, there is no limit, I can't see any limit situation.
- 90 (pause for a couple of minutes)
- 92 **STO:** Can you say something about what you are doing?
- 94 **Chris:** I am just sort of cross-checking with the task, by thinking differently. Now I have located a definition in the book, where they describe uniform continuity and then I am trying to make it fit. We have used topological arguments and not so much delta-epsilon and stuff like that. It seems as if we could solve the task by topological argumentation.
- 96 **Curt:** I am sitting and thinking a bit, and tried first to do..and couldn't really see how, well then I just have to try to go back and use non-topological arguments, try to construct some distances, try to take some elements and so on. But that's almost the same as you are doing, if you are trying to cross-check [yes].

Initially, it seems like Chris sketches the proof for uniform continuity (lines 83-85), but the words "*all deltas are smaller than one*", line 84, and the following dialogue between the students reveal that he is still thinking topologically. Chris wants to confirm his conclusion by "*checking*" it against another "*description*" of uniform continuity. Curt is also trying to build a non-topological argument. The two students then run into difficulties because if delta is less than one, they feel restricted to only looking at identical points:

- 102 **Curt:** Chris, don't we have a problem? [yes, I think so] because if it should apply for all epsilon, right, then there has to exist a delta such that for
- 104 all  $x$ 's and  $y$ 's then the distance from, from eh,  $f(x)$  to  $f(y)$  should be smaller than epsilon. But if epsilon for instance was smaller than one
- 106 .. oh yes, we are going into the non-dis, ah, I switched them, I thought we were entering the discrete metric, and then I thought, that doesn't work so well (erases something).
- 108 **Chris:** For instance, if it's arbitrary, oh yes. If it's arbitrary, discrete...
- 110 **Curt:** But we might have a problem anyway, because this epsilon distance, no, delta distance, it can't be, delta can't be less than one.
- 112 **Chris:** What do you mean?
- 114 **Curt:** Because if the distance between two points is smaller than one then it can only be the same point in the discrete metric, if you have two points with a distance smaller than one from each other in the discrete metric, then it's the same.
- 116 **Chris:** Yes, but yes then it's the same point. But the distance is not coming over. If you have your  $U$  over here, where the points are [that's true] where the points are not ... where the points have some distance, then when they enter the discrete metric, then they all have the distance one to each other. They are open, individually.
- 120

- 122 **Curt:** So the distance can never become bigger than one.  
**Chris:** No, it can't. It's one between all the points.
- 124 **Curt:** Yes, and then. (pause)  
**Curt:** Yes, but then it must be ...
- 126 **Chris:** I think so too. I actually think our topological argument is okay.  
**Curt:** But how? Try to justify it.
- 128 **Chris:** It's..our topological argument..it's that continuity, it's if the inverse  
 image of open sets is open, and then, in the discrete metric all sets are  
 130 open .. and closed. Yes, so if we have some, if we have an open set..  
**Curt:** Yes, but that is to prove continuity, but not to prove uniform conti-  
 132 nuity. You can not topologically, without taking distances into consi-  
 deration, you can't based on a set consideration, so far as I know.
- 134 **Chris:** Get to uniform continuity? There you have to have some kind of  
 distance? [yes] But it has to be okay, because no matter what epsilon  
 136 we get, the distance from  $f(x)$  til  $f(y)$  is less than epsilon, then the  
 distance, yes then there exists, no, there exists a delta such that the  
 138 distance from  $x$  to  $y$  is always smaller than that distance. What I am  
 trying to say? [I don't know] No, suddenly I don't either.
- 140 **Chris:** Now I'll just try to write down the definition.  
 (pause)
- 142 **Chris:** Yes, it must be uniform, you can just choose your delta to be some-  
 thing bigger than one [yes] then you get, no, that's not certain, that's  
 144 what you don't want to do.
- Curt:** But, but, it can't be. We don't have control over, we don't control  
 146 how far  $x$  and  $y$  .. that we know  $x$  and  $y$  in the domain does not  
 provide any information about how far  $x$  and  $y$  are from each other in  
 148 the codomain. That is, we can't control how big the movements are in  
 the codomain by controlling the movements in the domain.
- 150 **Chris:** As soon as we have a bit, we can't separate the points as soon as we  
 move away.
- 152 **Curt:** No, so it's not, if you allow, if you don't allow a delta bigger than  
 one, then every point is alone, then you can't take an  $x$  and a  $y$  with  
 154 a distance smaller than one. If you allow a distance bigger than one,  
 then you, then you don't control where the images of the points with a  
 156 distance bigger than one are from each other. They can be everywhere.  
 (12 text lines have been omitted)
- 168 **Chris:** I am still in doubts, even though ... but you can find, you can just  
 choose a delta as a half, but then you can only look at the same point.
- 170 **Curt:** But that's not good enough.

The difficulties seem to occur because they mix the definition of continuity in a general metric space with the definition related to the Euclidean space. In Curt's argumentation in lines 145-149, he relies on his statement image of the definition of continuity in  $\mathbb{R}^2$  with the usual metric (Euclidean distance). He does not realise that in the discrete space, if  $\delta < 1$ , there is only one point in the delta-ball around a given point, namely the point itself. At the end of their attempt to solve the task, Chris is able to give a very rough sketch of the proof (*"you can just choose delta as a half"*, line 168), but neither Chris nor Curt considers the argumentation to be satisfactory. Because I understood their misinterpretation



at this point in the process, I prioritised that they should move on to one of the other tasks.

#### 4.2.4 Team B

The two students in team B, Bill and Betty, read the task and Betty began by writing down a reproduction of the task formulation on her paper. Figure 4.3 contains the notes Betty made during the solution process. After three minutes go by, I encourage them to talk to one another.

$$\begin{aligned} \sigma_{\text{discret}} : \quad \sigma(x,y) &= \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases} \\ i(M, \sigma_{\text{discret}}) &= (M, \sigma_{\text{discret}}) = (M, \sigma_d) \\ \sigma_{\text{discret}} &\rightarrow \sigma_d \\ \sigma_{\text{discret}} &= \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases} = \sigma_d \end{aligned}$$

**Figure 4.3** Betty's notes from task 1. Her reproduction of the task has been omitted.

- Bill: Yes, yes, we just have to ...
- 2 Betty: ... read what it says.  
(pause, they both look in the textbook index)
- 4 Betty: Discrete metric, 291.
- Bill: Yes.
- 6 (pause for one minute)
- Bill: What do you think about that?
- 8 Betty: I am not thinking much. I am just trying to find out what an arbitrary metric is.
- 10 Bill: Yes, but ...
- Betty: This is just one like that?
- 12 Bill: Yes, it is.
- Betty: So  $\rho$  is a metric? Yes. On ... X?
- 14 Bill: Yes. I feel it's difficult to see what it is that we put in and what we get out.
- 16 Betty: Me too.

- 18 **Bill:** The function takes one element from one of the metric spaces, eh, with  
the discrete metric and carries it over to another metric space with  
20 some arbitrary metric. But the function is this identity. Which maps  
one element onto itself, right?
- Betty:** Yes. So the discrete metric and the arbitrary metric is the same?
- 22 **Bill:** Mmm.
- Betty:** If we take  $i(x)$  then we will get  $x$ . If we take  $i$  on that, the metric  
24 space, where ... what is that one called [the discrete metric] discrete,  
then we get a metric space where the metric is arbitrary.
- 26 **Bill:** Mmm.
- Betty:** I have difficulties understanding where continuity fits in.
- 28 **Bill:** But that's not what I have trouble with.
- Betty:** Okay. Try to explain to me what this is all about then.
- 30 **Bill:** The mapping is continuous, that's clear I guess. That something can  
be continuous, that's not what I mean. I mean, I can't understand  
32 how, how the mapping can do what it does. [Yeah] I mean if it maps  
one element onto itself how come it doesn't ... how come it doesn't  
34 take ... ?
- Betty:** That's because the discrete metric is a subset of the arbitrary metric.  
36 Or is contained in.
- Bill:** Yes, but ...
- 38 **Betty:** Continuous, uniformly continuous. (She laughs) Uniformly, this is  
when it's independent of  $x$ . No matter what  $x$  you choose, it follows,  
40 or then it's continuous.
- Bill:** Mmm.
- 42 **Betty:** Really, how would you interpret this bracket? (meaning  $(M, \sigma_{discrete})$ )
- Bill:** What bracket?
- 44 **Betty:** You have a set and on that set you have a metric, isn't it like that?
- Bill:** You have a space ...
- 46 **Betty:** Yes. ... (She reads in the textbook) 'A metric space is a set  $X$  with  
a function' [Mmm]. What is changing is then the metric? [Mmm] We  
48 go from the discrete to the arbitrary.

The students start discussing how they could interpret the two metrics. Betty reveals that she does not know the mathematical meaning of the word 'arbitrary' (lines 8-9). In mathematics, arbitrary is connected to generality. To look at an arbitrary entity of some kind (for instance a point, set, metric) means that this entity has precisely the same properties as all of the other entities of the same kind. Apparently, Betty at first sight thinks that an arbitrary metric is a specially defined metric, like the discrete metric. In the textbook, just opposite the definition of the discrete metric is the definition of a metric space. Betty spots this and asks Bill if it is "*just one like that*", meaning a metric satisfying the definition. Bill confirms this. It is clear that Betty is not really familiar with the definition because she asks Bill if  $\rho$  in the definition is a metric (line 13).

The following exchange (lines 17-26) shows why the two students find it difficult to interpret the task. They think that the identity function also maps metrics on metrics. On the one hand, they know how the identity works; it maps one element onto itself (it leaves something unchanged). On the other

hand, however, they can see that the metric changes. Betty tries to solve this conflict by suggesting that the discrete metric and the arbitrary metric are the same. Betty's confusion about how the identity works, transfers to difficulties in understanding how the situation could be related to continuity, but Bill is able to separate the two things. He understands that it is possible to ask if a function is continuous (lines 30-31), but he cannot see how the identity can change metrics (lines 31-34). Betty proposes another explanation founded on her understanding of sets: since  $\sigma_d$  is arbitrary, it 'contains' all metrics, also the discrete metric. The 'explanation', however, does not seem to resolve anything for them and lacking a strategy to provide clarification, Betty moves on to look at the requests in the task. Clearly, she does not have a precise comprehension of the difference between continuity and uniform continuity, although she has an idea about what the difference is technically (lines 38-40). Bill does not really respond and Betty returns to trying to clarify what the identity does – now focusing on creating meaning of the symbolism  $(M, \sigma_{discrete})$  (line 42), which only entails a recitation of the definition from the textbook (lines 46-47). Even though she actually reads that the metric is a function, she ends up concluding that the identity is responsible for the change of metrics.

Bill has skimmed through the chapter and found theorem 10.28, which characterises continuity in terms of converging sequences:

- Bill:** I am just wondering if we could use this one, 10.28?  
 50 **Betty:** How? I don't recognise anything. I can't see, I can't recognise anything in it.  
 52 **Bill:** No.  
**Betty:** I think maybe we should use 10.51 and 10.52. Uniform continuity,  
 54 because it says something about if  $f$ , if  $f$  is uniformly continuous on  $E$ , if and only if  $f$  is continuous on  $E$ . But  $f$  is not continuous?  
 56 **Bill:** But that is what we are suppose to determine.  
**Betty:** Or  $i$  it's called.  
 58 **Bill:** We can only use this afterwards, right?

This passage is dominated by directionless exploration in which they try to find applicable statements in the textbook. Betty indicates that she is basing her selection of relevant statements on identifying similarities between the task and the statements (lines 50-51). She finds some statements that include both continuity and uniform continuity, but Bill concludes that they can only use them in the second part of the task. During their exchange, Betty reveals that she has already determined that the identity is not continuous, explaining her arguments as follows:

- Betty:** Yes, but, if it's the discrete metric, then it must jump between zero  
 60 and one, right? Then it's not particularly continuous. Or what?  
**Bill:** Eh, but, this is not the function you should be looking at. It's not the  
 62 mapping. It's not  $\sigma$ .  
**Betty:** But what is it then?

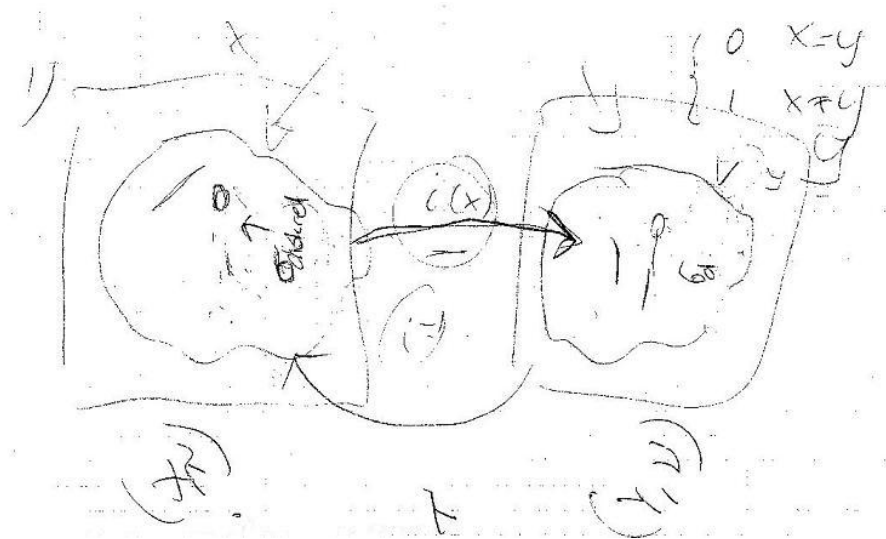
- 64 **Bill:** It's  $i$  that has to be continuous.  
**Betty:** Yes ...?  
66 **Bill:** So it just takes one element from one metric space and puts it in another one.  
68 **Betty:** Yes.  
**Bill:** But it's just about checking if this situation satisfies definition 10.27.  
70 (They both look at 10.27)  
**Bill:** But I can't use that.  
72 **Betty:** Mmm.  
**Bill:** What you said before. One of the spaces, is that a subset of the other  
74 space?  
**Betty:** Yes, I think so. I don't know if it's a subset, but. Yes it, or is  
76 contained in the other.  
**Bill:** Yes.  
78 **Betty:** If you have  $i$  and that, right, then it should be itself, because it's the identity, right [Mmm]. But we know that we get this one, so that  
80 implies that this is equal to that. Or this is a part of that?  
**Bill:** Yes, but it ...  
82 **Bill:** Can you in any way put this into that? (He looks at the definition 10.27)  
84 **Betty:** But what is that  $\tau$ -function? Is it just some metric on ...? Two function values, no ...  
86 **Bill:** Isn't it some kind of topology? Or what is it suppose to be?  
**Betty:** It's so frustrating that we have skipped this, I think. (They browse  
88 through the textbook)  
**STO:** I think you should proceed to task three.

Bill finally determines that the question of continuity can be settled by examining if the situation satisfies the epsilon-delta definition (definition 10.27), but he cannot figure out how to apply the definition (lines 69-71). It is not clear what is making it difficult for him. Bill remembers something Betty said about the discrete metric being a subset of the arbitrary metric (line 73). Bill is assuming that she is talking about the spaces, but it seems that Betty is not sure if she is actually referring to the metrics themselves or the metric spaces. Bill wants to insert their interpretation in the definition (line 82), but neither of them are apparently familiar with the definition (lines 84-86) and Betty even says that they have never discussed the definition in class (line 87). This might actually be true. Michael, the professor, was absent the day section 10.2 (where the definition is placed) was on the syllabus. Another professor substituted, but he did not mention the definition and Michael did not summarise the chapter when he returned. Since the two students were unfamiliar with the definition and were unable to interpret the symbols, I judged that it was unlikely that they would progress further, so I asked them to move on to another task.

#### 4.2.5 Team A

Like the students in team B, team A, which consists of Alan and Anna, also starts by examining the individual notions appearing in the task. Figure 4.4

contains Alan's notes. After having read the task for one minute, Alan begins the discussion.



**Figure 4.4** Alan's notes from task 1.

- Alan:** Okay, I have read it. (They both laugh) Well, the arbitrary eh metric, is that the opposite of the discrete? Are those each other's opposites? I can't remember that I have heard about the arbitrary?
- Anna:** But an arbitrary metric, that is just some random one.
- Alan:** Yes, that is just everything, right? [yes] And the other one, that is the one which goes to zero and one.
- Anna:** What other one?
- Alan:** The discrete one.
- Anna:** It has another name, doesn't it?
- Alan:** Yes, but we are not taking out the books to find out.
- Anna:** No, no, we are not. (They laugh)
- Alan:** No, we won't.
- Anna:** Okay.
- Alan:** (He writes in the upper right corner of the paper) This is the one that looks like that and has something and then it goes to zero when something that does something up here ... I can't remember just now, it depends on .. but it goes to zero and one, the discrete one.
- Anna:** What is that one called which determines if it's in or outside the set? It's called something, what is it called? It's also zero and one it goes to. It's annoying, it's right on my tongue, I know what it's called. Well, it's not relevant.
- STO:** Indicator function.
- Alan:** Indicator.
- Anna:** Precisely. The indicator function.
- Alan:** Yes.
- Anna:** Well.

The opening dialogue between the students concerns a clarification of the two metrics. As in Betty's case, Alan also reveals in his first comment that he does not know what 'arbitrary' means in mathematics. He perceives it as a name for a specific metric (talks about it in the definite form) and tries to interpret it as the "*opposite*" of the discrete metric (whatever he means by that). In her response, Anna shows that she can interpret 'arbitrary' in the everyday meaning of the word (as something random), but she does not clarify the mathematical meaning. It is of course not possible to conclude that she is unaware of the mathematical meaning, but since she chooses to give an everyday interpretation of the word, she presumably does not know that arbitrariness in mathematics is connected to generality. And Alan is not able to be more specific; in fact, he is even more imprecise ("*Yes, that is just everything, right?*"). The short dialogue does not end with a clarification that an arbitrary metric is one which satisfies the three conditions stated in the definition of a metric. Next, they move on to the discrete metric. Alan describes it as something "*going to zero or one*", and writes the symbols:

$$\begin{cases} 0 \\ 1 \end{cases}$$

in the upper right corner of his paper (see figure 4.4).

Anna gets an association, and although she is aware that it is irrelevant, she cannot put it out of her mind (lines 18-21). As a result, I decide to interrupt and terminate her association process, and provide the name of the indicator function. At this point in the process, Anna has only obtained a recollection of what the indicator function does. Alan has ascertained that the arbitrary metric is "*random*" and "*can do everything*". In addition, he has connected the discrete metric with 0 and 1, although without being aware of the conditions. In general, they have not managed to obtain any information or knowledge useful for reaching a solution. Anna returns to the requests in the task, and Alan starts to make an illustration, still without having clarified what the discrete metric actually does:

- (10 sec. pause)
- 28     **Anna:** We are supposed to decide if the mapping is continuous. And uniformly continuous.
- 30     **Alan:** What if we make a sketch of it? That is always good [yes]. Then we have one of these called  $(M, \sigma_{discrete})$  and this must then consist of zeroes and ones.
- 32     **Anna:** I am sorry, but I am going to look in my notes (She is referring to the notes on measure and integration theory). ...
- 34     **Alan:** I don't think it's the notes so much, I actually think it's more ...
- 36     chapter 10 you should use.
- Anna:** Do you?
- 38     **Alan:** Then there is an arrow here.
- (20 sec. pause)

- 40 **Anna:** Do we agree that no matter what we operate on here (presumably  
the domain), it's the identity we have, so we'll get the same over here  
42 (presumably the codomain of the identity function).
- Alan:** Hmm, that's what I ... over here (presumably the domain) there must  
44 be zeroes and ones because the discrete metric is nothing else, it can  
only make that. And this is the one that is carried over to the other  
46 called  $(M, \sigma_d)$ .
- Anna:** Yes, but it's still the identity we are dealing with.
- 48 **Alan:** Yes.
- Anna:** So no matter what we insert we get the same out again.
- 50 **Alan:** Yes.  
(8 sec. pause)
- 52 **Anna:** I don't feel I really know these notes (7 sec. pause). Do you take  
chapter 8, or sorry, 10 [10, yes].
- 54 (They look in the notes and the textbook, 15 sec.)
- Alan:** It's supposed to be here ... oh, here it was. (10 sec. pause) The  
56 discrete metric, this is the one you use to find out if two points are  
identical, right? Isn't that the whole point with it? [Yes, it is] (Alan  
58 refines the definition in the upper right corner) Yes, and this is of course  
equal to  $y$  and different from  $y$ , and that is, that's what I couldn't  
60 remember, right. And this is, isn't this the identity, this is sort of the  
function here, no, what is it that does this? (presumably he means the  
62 arrow between the two spaces he has been drawing).
- Anna:** It's  $i$ .
- 64 **Alan:** Is it  $i$ ?
- Anna:** Yes, it's the mapping  $i$ .
- 66 **Alan:** It takes this to that.
- Anna:** What do we know about the discrete metric?
- 68 (12 sec. pause)
- Alan:** Well, we know that it only has two values.
- 70 **Anna:** Yes, but we have to know something else.  
(30 sec. pause)
- 72 **Alan:** Well, it stands here, this was what I was looking for. (He laughs)
- Anna:** What did you look for?
- 74 **Alan:** If it was in fact  $i$ . It was nothing. Well, but can't we say,  $i$  then takes  
for instance a zero and takes it over. And then it happens..
- 76 **Anna:** The worst part with these kinds of tasks is, that I am convinced  
that this is some of the easiest you can meet. But I have a hard time  
78 figuring out what we know and how we can use it once we realise what  
we know.
- 80 **Alan:** Yes.

In line 31, Alan assumes that the discrete metric in some way determines which elements belong to  $M$ . It seems as if he equates the range of  $\sigma_{discrete}$  with  $M$ , but this is not a consistent misinterpretation since he states that  $M$  consists of zeroes and ones (in plural). In the meantime, Anna is both trying to look in her notes (on measure and integration theory) to find clarification and to help Alan make an illustration and interpretation of the task. In line 40, she starts to focus on the different entities in the task; to begin with, she concentrates on the identity function. Alan, on the other hand, focuses on the discrete metric

and the elements in  $M$ , which means they are not paying attention to the same aspects of the task. Alan repeats that the elements of  $M$  are determined by the discrete metric and that they consist of zeroes and ones, because “*it can only make that*” (line 44). At this point he has not yet clarified the definition of the metric. He continues to explain “*And this is the one that is carried over to the other called  $(M, \sigma_d)$* ”. By “*this is the one*”, Alan presumably refers to the metric space  $(M, \sigma_{discrete})$ , so based on his illustration (where the underlying sets of the domain and codomain of  $i$  are depicted as two different sets) and his description, Alan has made it confirmative that the underlying sets of the domain and range are different. This induces a conflict in Anna (line 47). She knows how the identity function works, “*no matter what we insert we get the same out again*”, meaning that if they ‘insert’ the metric space  $(M, \sigma_{discrete})$ , they should get  $(M, \sigma_{discrete})$  and not  $(M, \sigma_d)$ . She momentarily solves or postpones her internal dilemma by searching for help in the notes, and she persuades Alan to look in the textbook. Although it is not apparent whether they know what they are looking for, Alan finally finds the definition of the discrete metric (line 55) and manages to infer the conditions on his paper (see figure 4.4):

$$\begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

He remembers or interprets that the point of the discrete metric is “*to find out if two points are identical*”. They now have two functions which could be related: *the identity*, which maps an element onto itself, and the discrete metric, which finds out if two elements are *identical*. The students’ descriptions of the two functions are very similar, so it is not surprising that they find it hard to distinguish between them. This could explain Alan’s confusion about what the arrow in his drawing symbolises (lines 60-62). Is it really the identity? Anna confirms this, but without providing any explanation.

Having trouble sorting out what is going on, Anna suggests that more information must exist about the discrete metric that they can use to clarify the situation. Alan reveals that he knows that the discrete metric only has *two* values, one and zero (line 69). Alan, on the one hand, knows that the discrete metric only ‘produces’ two numbers, while, on the other, he believes that the domain consists of several zeroes and ones. A very reasonable possibility is of course that Alan is being unclear about his interpretation and that he believes that the range of  $\sigma_{discrete}$  has two elements, while the set  $M$  has more than two elements. Another possibility is that Alan’s has a more complicated (mis)interpretation of the task:  $M$  is a set of many elements (not necessarily ones or zeroes) upon which the discrete metric acts such that every element in  $M$  is replaced by either zero or one. This would explain the many zeroes and ones in  $M$ . But this entails that the discrete metric is a one-variable function. Where does Alan get this image from? One possibility is that the talk about a function going to one or zero, which made Anna think of the indicator function, had a negative influence on



the clarification of the task situation. The indicator function is exactly a one-variable function which ascribes the numbers one or zero to elements (whether they belong to a certain subset or not). Consequently, what appeared simply as a harmless, although time-consuming, distraction might be responsible for some of the students' misconceptions.

Anna's frustration about not being able to clarify the situation is expressed in her comments starting in line 76. True, they have acquired some knowledge, for instance, that the discrete metric "*finds out if two numbers are identical*", but they do not know how to use this knowledge constructively. After trying to clarify the set-up, it is again Anna who returns to the request:

- 82       **Anna:** Well, if we could say something about if this one is continuous and  
if it's uniformly continuous.
- 84       **Alan:** Can it really be that if it's only zeroes and ones? [I don't think so]  
And then it's something about the inverse image over here and if it  
goes back and this fits then everything comes together, eh.
- 86       (5 sec. pause)
- 88       **Anna:** But how can we understand this? Because if it's zero and one if two  
point are equal or different, what do we then get over here?
- 90       **Alan:** Yes, but that is this, eh, this identity, that was what I was thinking  
about.
- 92       **Anna:** Yes, but no matter what number you put into the identity, you get  
the same number out again.
- 94       **Alan:** I would say, I get zeroes and ones over here also, that is what you  
say?
- 96       **Anna:** Yes, but I am not certain that it gives the same. Because I can't see,  
if this is an arbitrary metric with some ...
- 98       **Alan:** But if you say that it's born over there, then it can only make the  
same over here.
- 100       **Anna:** Yes, that is what it must do.
- 102       **Alan:** The identity, shouldn't we be sure that it actually does what we think?  
(Anna laughs)
- 104       **Anna:** It does (pauses for 15 sec.) it says so in the task, Alan.
- 106       **Alan:** Does it say that? Oh, yes, it's right there. (He laughs) But a zero  
and a one, can one say that lying in a metric space, that it's especially  
continuous? Can't we say, that from logical considerations, that ...
- 108       **Anna:** Just say that that is how it is? (She laughs)
- 110       **Alan:** It can't be continuous, this over there, I think?
- 112       **Anna:** No, but this is not the one which has to be continuous, it is the  
mapping which has to be continuous.
- 114       **Alan:** Yes, that's right.  
(9 sec. pause)
- 116       **Anna:** If this was at the exam, I would begin to browse through the text-  
book directionlessly to find out what does it mean that a mapping is  
continuous.
- 118       **Alan:** Yes, we can say that it's both one-to-one and bijective, and it must  
be onto .. yes then it's onto, right (They laugh). It is, right? Because  
it take a point over here and throws it over there and that point over  
there.

120 **Anna:** It is one-to-one, but it isn't onto because there are probably a lot more numbers over here than zero and one.

**Alan:** That's right. This over here, that's arbitrary, that's right. Yes.

In this excerpt, the students' concept image of continuity is expressed. Alan questions whether 'something' that only consists of zeroes and ones can be continuous, and Anna share his doubts. As before, when Alan tries to explain the situation, he uses many pronouns ("*can it even be that when this is only zeroes and ones?*"), which makes it difficult to know what he is referring to, and what is worse, it obliterates the differences between the concepts in play in the task and complicates the clarification process for the students. Out of the blue, Alan recollects parts of the topological definition (lines 84-85), but Anna seems to be too confused to pay any attention to this. She is still trying to make sense of the situation. She knows that the identity maps one element to itself (line 91), and thus does not change the set upon which it is defined. She is convinced, however, that the domain and the codomain are different, because the two metrics are different (line 95); somehow, the identity is able to change the metrics.

Alan senses her conflict and presents a 'reasonable' solution to her confusion (line 100): they have misunderstood what the identity function actually does. Now convinced that this is not the case (where he only considers the output of the function, not the domain of the function), he returns to considering how the identity can be continuous when *it* consists of zeroes and ones (lines 103-105). Clearly, his concept image of a continuous function is highly influenced by the concept of a continuum: a function defined on a set which is not a continuum cannot be expected to be continuous at all, and since the identity apparently is defined on a set of zeroes and ones, it cannot be continuous. Anna clearly believes that he is talking about the discrete metric and not the identity ("*No, but this is not the function which has to be continuous*"), and Alan agrees with her. Nevertheless, what looks like a potential breakthrough in the clarification process does not give rise to any reformulation of the situation.

Anna's next comment is interesting (line 112) because it is the first time that either of the students mentions the possibility of clarifying what it means to be continuous. This could have been a beneficial strategy, but for some unknown reason, Anna ranks it alongside "*browsing through the textbook*" without having a plan or direction, a strategy that the two students several times have described as unacceptable (e.g. lines 10-11 and line 33). She dismisses the strategy.

122 (13 sec. pause)

124 **Anna:** Hold on, wait a minute. Those two are metric spaces and on these two metric spaces we have some  $\sigma$ , over here it's  $\sigma_{discrete}$  and over here it's  $\sigma_d$ . Are we agreeing on that?

126 **Alan:** You sound very secretive. Something is about to emerge. I can feel it.

128 **Anna:** No, not at all. I just think that something is about to fall into place. It isn't these two (meaning the two metrics, supposedly) that we are

- 130 interested in. If this is zero or one or if this is zero or one or whatever,  
that doesn't matter. We are interested in  $i$  [Yes, that is it] So no matter  
132 [what we pour from this side (the domain)] This can consist of a lot  
more than zero and ones, can't it? We just have this  $\sigma$  working on our  
134 metric spaces and it finds out if two numbers are equal or not.
- Alan:** But what if this over here is something connected, here is something  
136 connected (in the domain), then this mapping, then this over here  
should also be something connected, then you say that the mapping is  
138 [continuous] is continuous, right?
- Anna:** Yes.
- Alan:** And this over here, this isn't especially connected.
- Anna:** Yes, but this zero and one this is just the result of this one [yes,  
142 yes]. Over here you could have all the real numbers [yes, but that's  
right]. And also all the complex numbers, and over here you also have  
144 something that gives some kind of result. But no matter ... if we take  
some number over here (in the domain) then it should be the same  
146 number over here (the range). This is the identity.
- Alan:** So you claim from this point of view that it's continuous even though  
148 the result is all chopped up?
- Anna:** What do you mean by chopped up?
- Alan:** Those two points are not continuous, if they are in the set, because  
150 there is nothing in between.
- Anna:** But I just think that what we are sitting here and wondering about,  
152 the zeroes and ones from the discrete metric. I don't think that that's  
what we should be thinking about, we ought to think about it all. It's  
154 just that the discrete ...
- Alan:** It could be anything, that one, some collection of something.
- Anna:** We could have the numbers from one to ten, yes. And the only thing  
158 it does, is to take two numbers and determines if they are identical.
- Alan:** Yes.
- Anna:** This one over here (the arbitrary metric), it can do anything, it can  
160 say ...
- Alan:** Yes, yes, then it's true, then it does, then it does, then it's continuous,  
162 that is then it does it ... it's for all, right?
- Anna:** Yes. What we have to be sure about is then ...
- Alan:** Let's see what the textbook says about continuity.  
166 (8 sec. pause)
- Anna:** I think I have to go home and organise these notes.  
168 (1 min. pause)
- Anna:** Now the question is if we should move on to another task and then  
170 go back to this one if this is..if we are allowed to do that.
- Alan:** But it satisfies the definition of continuity, I think.  
172 (10 sec. pause)
- Anna:** What page is it? 299.  
174 (30 sec. pause)
- Betty:** But we can't really say anything based on this definition, can we?
- 176 **Alan:** No, and all the time I sit here and think about balls and stuff like  
that and I don't really feel that ...

In a moment of enlightenment, Anna realises that the underlying set of the domain and codomain,  $M$ , can be separated from the metrics, but she does

not reach an explanation or interpretation of what the metrics then do if they do not determine the elements of the underlying set of the domain and the range. Alan does not understand what Anna means. Instead, he restates his concept image of continuity, now promoting it as the officially accepted definition (the underlined part of the sentence): “*here is something connected (in the domain), then this mapping, then this over here should also be something connected, then you say that the mapping is continuous*” (lines 135-138). This shows that his concept image has now become his concept definition. Anna agrees with him, but insists on the fact that the domain is not necessarily *non-connected*, and Alan agrees that  $M$  in the domain could in fact contain the real numbers and maybe even the complex numbers, but that this does not contradict his beliefs about what the discrete metric does. It still attaches a zero or a one to each element in  $M$  before the identity maps the numbers. Anna has difficulties understanding him and tries to maintain focus when Alan suddenly sees a new alternative: since  $\sigma_d$  is arbitrary, it “*can do anything*”, even changing something “*all chopped off*” into something connected, “*Yes, yes, then it’s true, then it does, then it does, then it’s continuous, that is then it does it ... it’s for all, right?*”, where the underlined pronouns supposedly refer to  $\sigma_d$ . Since “*it’s for all*”, he can choose one of the metrics that can convert something ‘chopped off’ into something connected. Anna agrees with him (line 164). At first glance, Alan finds that the new interpretation satisfies the definition of continuity. Anna also looks up the definition, but after having studied it neither of them thinks it is possible to use it.

Even though the two students try to listen to each other and comment each others’ ideas and suggestions, it is nonetheless clear that their way of working together hinders a constructive solution process. Most often it is Anna who manages to reason in a potentially constructive way, and Alan who holds on to incorrect interpretations and faulty concept images, thereby hindering constructive attempts to solve the task.

The presentation of the process stops here, but the two students continue to try to solve the task. I, however, feel that most of the important and characteristic elements of their process and their mathematical understanding of the concepts involved have been introduced.

#### 4.2.6 Combining the four solution processes

Besides comparing the four solution processes (of task 1), this section also summarises what I view as the main reasons for the observed solving difficulties. In chapter 7, this summary is also incorporated in answering the first research question.

##### 4.2.6.1 Was the task a problem?

At least one of two aspects made the task a problem for (presumably) each of the eight students. The first aspect is related to the content of the task, while

the second concerns the notation used. When reviewing all the tasks that the professor has assigned during the course, none of the tasks asks the students to justify whether a specific function between two different metrics is continuous or not. This suggests that the students have not practised using the definition of continuity in these kinds of tasks. This assumption is underpinned by Danny's insight about the clues involved in the task *"that continuity both depends on the metric in the space you come from and in the space you are going to."* This is not something he has learned in class, but something he discovers for the first time when reading the task.

The processes of teams A and B reveal that the notation  $i : (M, \sigma_{discrete}) \rightarrow (M, \sigma_d)$  contributed to their difficulties in clarifying what the identity function actually does. The textbook does not use the notation  $(set, metric)$  to represent a metric space. In fact, it uses symbols such as  $X$  and  $Y$  to indicate both sets and metric spaces. This is justified by a comment placed on the page right after the definition of a metric space where the author writes that  $X$  and  $Y$ , which until this point in the textbook have represented sets, from now on *"will represent arbitrary metric spaces (with respective metrics  $\rho$  and  $\tau$ )." [Wade, 2004, p. 291].* Readers are expected to remember this comment when they read the subsequent definitions and theorems. For instance, in the definition of continuity it is not explicitly stated that  $\rho$  and  $\tau$  are metrics. Since the students in team C have also used the textbook [Carothers, 2000] to prepare for the exam, and this book uses the notation  $(set, metric)$  to represent a metric space, it is highly likely that they are familiar with this notation.

#### 4.2.6.2 Similarities and differences

The first stage of the problem solving proceeds very differently in each of the four teams when they encounter a problematic situation. Teams C and D immediately or very quickly reach a formulation of the main point in the task, whereas teams A and B struggle to make sense of the situation in the task. At first sight, it seems that teams C and D are able to skip the stage of clarification, because they immediately understand the task. Team D moves straight to a formulation of the main point, while team C discusses a strategy. It is not possible to determine what Danny (team D) is thinking about in the first twenty seconds of reading the task before he formulates the main point; maybe he is wondering about the notation and why the metrics are different, but it is not possible to say. After a closer look at the process in team C, it seems that a stage of clarification actually takes place. After having talked about a strategy, the students separately try to clarify how the identity function works. Like the students in teams A and B, they perhaps also experience discord about how the identity could induce a change of metrics. As the subsequent exchange in team C demonstrates, both students are able to get through the clarification stage successfully, but it is not clear what makes this possible. Teams A and B are stuck in this stage because they do not possess the necessary mathematical resources (as Schoenfeld [1985] defines it; see

page 58) to interpret the task correctly.

When the students in teams A and B finally decide on a solving strategy, it becomes clear that they are unable to interpret and use the definition of continuity in a metric space setting. The difficulties involved in interpreting the task and making sense of and using the definition of continuity are naturally related. They are the result of a collection of difficulties relating to learning mathematics, including: inadequate concept images; a lack of knowledge about functions, mathematical notation and symbolism; a lack of precision and accuracy in both speech and writing; an inability to incorporate insights gained; a bias toward the specific (as opposed to the general); and a lack of training in constructing epsilon-delta proofs, just to mention a few.

From my point of view these causes can be divided in two groups. One group concerns knowledge about and understanding of notions related to mathematical structure (definition of concepts, relations between concepts etc.), while the other concerns mathematical rules, both written and unwritten, and details (notation, precision, proof strategies etc.). These causes are related in a non-trivial way. For instance, difficulties understanding the concept of functions are related to difficulties understanding the symbolism associated with this concept (this became clear in the solution processes of teams A and B). Hence, difficulties related to understanding the mathematical structure are coupled to difficulties related to understanding mathematical details and vice versa.

#### 4.2.6.3 The interplay between structural and detailed focus

Teams A and B go through many of the same difficulties but not in the same order. After clarifying the two metrics, team B experiences confusion about how the identity can change metrics. In team B, the situation gets more complicated, because the metric space in some way is believed to ‘produce’ the elements of  $M$  (the indicator function presumably plays a part in this misinterpretation). But the students in this team also struggle with the ‘fact’ that the identity must be able to change metrics.

Why does this confusion occur and why are the students in teams A and B unable to construct an explanation that makes sense? First of all, the students have a very limited concept image of a metric space. Alan, who is familiar with the discrete metric, has an image of it as something that determines if two numbers are identical, but none of the four students seem to have an image of a metric as something that determines the structure of a space and influences properties of functions such as continuity. At some point, Betty locates the definition of a metric space, but she does not know how to apply the information usefully. For instance, she could have become aware of the detailed properties of a metric: a metric defined on  $M$  is a function from the product space  $M \times M$  into  $\mathbb{R}$ . Compared to the definition of the identity function,  $i(x) = x, x \in M$ , this might have led to the conclusion that  $i$  could not operate on  $\sigma_{discrete}$ , since ‘ $\sigma_{discrete}(x, y)$ ’  $\notin M$ . Also in team A, the omission of important details

might prevent clarification. When Alan writes down an amputated version of the definition of the discrete metric, he neglects to write ' $\sigma(x, y) =$ '. He then fails to include important information that could have led them to reconsider their interpretation. Although in this case, while Alan uses the discrete metric to construct elements of  $M$ , it would be more demanding for him to realise that the range of  $\sigma_{discrete}$ , the set  $\{0, 1\}$ , and  $M$  are not necessarily the same. Compared with the process of team D, Danny is very careful to provide the correct details. He explicitly says to Dylan that it is important to write down the definition of continuity correctly. When he writes down the definition of the discrete metric he is careful to specify that it is a function of two variables (Betty also specifies this on her paper, see figure 4.3, but she copies from the textbook so it is likely that she did not pay attention to this detail).

Another inability to make sense of mathematics can be seen in teams A and B's attempts to implement the strategy chosen. Presumably, since the symbol  $\tau$  does not appear in the task, Betty is confused (the symbol  $\rho$  is used in the definition of a metric space, which Betty noticed in the beginning of the solution process, but  $\tau$  is not mentioned) and Bill suggests that it could represent a topology, probably because  $\tau$  represents a topology in the measure and integration notes. It is not clear why Anna rejects the applicability of the definition when Alan introduces it. Maybe she dismisses it after a shallow comparison with the task and finds inconsistencies between the symbols in the task ( $\sigma_{discret}$  and  $\sigma_d$ ) and in the definition ( $\rho$  and  $\tau$ ). Another possibility is that since the textbook definition apparently does not coincide with Alan's own definition of continuity (a short version of his 'definition' could be: 'a continuous function maps connected sets onto connected sets'), the textbook definition is viewed as useless. The latter explanation leads to the conclusion that erroneous concept images<sup>5</sup> can foster erroneous concept definition images, which prevent the student from applying the correct definitions.

This conclusion is substantiated by the solution processes of teams A and B when they attempted to solve task 4. In this case, they are asked to determine whether the collection of intervals on  $\mathbb{R}$  is a sigma-algebra. When they read the task they immediately seem convinced that they need to check if the three conditions in the definition are satisfied. Nothing in the dialogues suggests that either of the two students have created a strong concept image of a sigma-algebra. In fact, neither of them seems to know the purpose of introducing the concept and they have never been introduced to a collection of sets which is not a sigma-algebra. Consequently, it is perceived as an odd mathematical construct without practical importance defined by the three conditions. It thus seems that in the absence of concept images, weaker students may appear more able to choose and implement a strategy in which they check the fulfilment of a definition than when

<sup>5</sup> Alan's concept image of a continuous function of course originates from the concept image learned in high schools, where continuity is translated with connectedness: a function is continuous if the graph of the function is connected [Nielsen & Fogh, 2006, p. 62].

they have develop an erroneous or incomplete concept image.

Before carrying out the study of the students' solving processes, I wondered whether the framework of Lithner [2003] for characterising students' reasoning could be applied when the tasks include proof construction. All four teams at some point determine that it is a matter of checking whether the situation satisfies the definition; none of them justifies this choice of strategy. Therefore, it is rather difficult to characterise their reasoning according to the three categories (plausible reasoning, reasoning based on identified similarities or reasoning based on established experiences) or to the categories in Lithner's later published framework (memorised reasoning, algorithmic reasoning, creative mathematically founded reasoning) [Lithner, 2008].

There are, however, some differences in how the students reason before coming up with the strategy. In team C, Chris and Curt discuss the situation in terms of the *content* of the topological definition of continuity, which is also the case in team D. In both teams, the students concentrate on what is pivotal in the determination of continuity. Team C discovers that it is examining whether openness changes when the metric changes, while Danny notices that continuity depends on both metrics. Neither team worries about whether the function is actually continuous or not. In teams A and B, the choice of strategy stems from discussions in which the students try to determine – by using their limited concept image – whether the function is actually continuous or not. Mixed in with the discussion is an attempt to find out whether the relevant function is  $\sigma_{discrete}$  or  $i$ .

In addition, the opinion about the usefulness of applying the definition is different in the two groups of teams. In teams C and D, the students express no doubts about the appropriateness of using the definitions, and both teams are capable of handling the definitions. In teams A and B, the students have trouble making sense of and using the definition of continuity. When the students in team A cannot make sense of the definition, they merely dismiss this approach, and thus do not view the solving strategy 'use the definition' as a solving approach that can always be applied (although it might not be the most expedient) when solving a mathematical task. In team B, Bill tries on two occasions to interpret the symbols, but gives up.

Betty's final comment that they did not talk about the definition in class communicates a variety of different and interesting information. It shows that Betty has followed the course conscientiously, because she is capable of determining that they have skipped the definition (others might conclude that they were absent that particular day). The definition was scheduled for the second lesson so either Betty has a very good memory or she experienced several times during the course that she was unable to manage applying the definition. The second possibility underpins the observation that she, as well as Anna, Alan and Bill, is unable to construct meaning of new mathematics without being guided. This substantiates my suspicion that the two different types of solving difficulties experienced by the students (making sense of the problematic situation and



being able to apply the definition) are both connected to overall difficulties with constructing the meaning of new or forgotten mathematics.

#### 4.2.7 Task 2

The difficulties experienced by the students in teams A and B are dominated by a lack of sufficient mathematical resources ([Schoenfeld, 1985]), while this is not the case for teams C and D. In order to supplement the example base, I present the solution process during which team C attempts to solve task 2, but fails to construct the proof. It will be clear from the presentation that the two students possess the mathematical resources specifically needed to solve the task, or at least insufficient resources is not the main reason for their difficulties (which is also the impression the team's three other solution processes give).

##### Task 2

Assume that for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  there exists a constant  $K$ , such that  $|f(x) - f(y)| \leq K|x - y|$ , for all  $x, y \in \mathbb{R}$ . Show that  $m^*(f(E)) \leq Km^*(E)$ , for all  $E \subseteq \mathbb{R}$ , where  $m^*$  is the outer Lebesgue measure.

Since the solution process of team C does not reveal how the task can be solved, I provide a proof of the statement. The numbers in square brackets will be used in a later analysis of the proof. In the proof,  $\ell(I) = b - a$  stands for the length of an interval,  $I$ , of the form  $[a, b]$ ,  $]a, b[$ ,  $[a, b[$ , or  $]a, b]$ . Propositions 16.1 and 16.4, referred to in the proof, are listed in appendix C.2.

##### Proof

[1] First we look at the special case where  $E = ]a, b[$ . Since  $f$  is Lipschitz<sup>6</sup>,  $f$  is uniformly continuous, and it is possible to define the extended function  $\bar{f}$  on the closure of  $E$ , i.e.  $\bar{f} : \bar{E} \rightarrow \mathbb{R}$ , where  $\bar{f}(x) = f(x)$ ,  $x \in ]a, b[$ ,  $\bar{f}(a) = \lim_{x \rightarrow a+} f(x) < \infty$  and  $\bar{f}(b) = \lim_{x \rightarrow b-} f(x) < \infty$ . Since  $\bar{E}$  is closed and bounded and  $\bar{f}$  is continuous, there exist points  $x_m, x_M \in \bar{E}$  such that  $\bar{f}(x_M)$  and  $\bar{f}(x_m)$  are the maximum and minimum, respectively, of  $\bar{f}$  on  $[a, b]$  and  $\bar{f}(\bar{E}) = [\bar{f}(x_m), \bar{f}(x_M)]$ . Then the following holds:

$$\ell(f(E)) = \ell(\bar{f}(\bar{E})) = |\bar{f}(x_M) - \bar{f}(x_m)| \leq K|x_M - x_m| \leq K|b - a| = K\ell(E).$$

Since  $m^*(I) = \ell(I)$  for any interval  $I$  (proposition 16.4), the inequality  $m^*(f(I)) \leq Km^*(I)$  is satisfied for any interval  $I$ . [2] Let  $E$  be any subset of  $\mathbb{R}$ . Without loss of generality, assume that  $I_n$  and  $J_i$  are sequences of disjoint intervals such that  $E \subset \bigcup_{n=1}^{\infty} I_n$ , and  $f(E) \subset \bigcup_{i=1}^{\infty} J_i$ . [3] Since  $f$  is uniformly continuous, the image of an interval is also an interval, and the image of the covering of  $E$ ,  $\bigcup_{n=1}^{\infty} f(I_n)$

<sup>6</sup> A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $|f(x) - f(y)| \leq K|x - y|$ , for all  $x, y \in \mathbb{R}$  is said to be *K-Lipschitz*.

will also be a covering of  $f(E)$  (the intervals  $f(I_n)$  may not be disjoint). [4] The outer Lebesgue measure of  $f(E)$  then becomes:

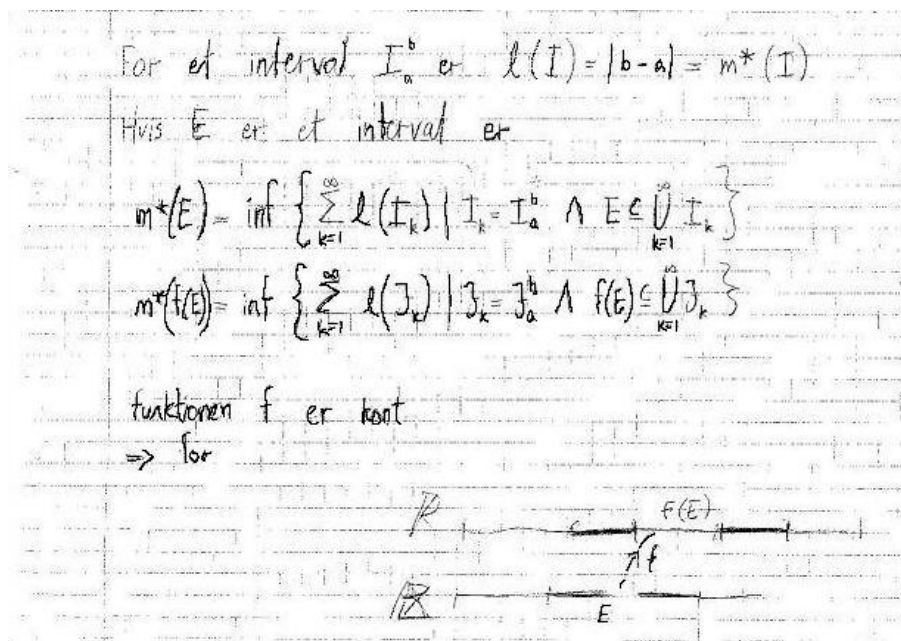
$$\begin{aligned}
 m^*(f(E)) &\equiv \inf \left\{ \sum_{i=1}^{\infty} \ell(J_i) : f(E) \subset \bigcup_{i=1}^{\infty} J_i \right\} \\
 &\leq \inf \left\{ \sum_{n=1}^{\infty} \ell(f(I_n)) : E \subset \bigcup_{n=1}^{\infty} I_n \right\} \\
 &\leq \inf \left\{ \sum_{n=1}^{\infty} K \ell(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n \right\} \\
 &= K \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n \right\} \\
 &= K m^*(E),
 \end{aligned}$$

where the infimum is taken over all coverings (of disjoint intervals) of  $f(E)$  and  $E$ , respectively. The first inequality is due to the fact that the intervals  $(f(I_n))$  might not be disjoint (proposition 16.1), and the second inequality follows from the derived inequality  $\ell(f(I)) \leq K \ell(I)$ , where  $I$  is any interval in  $\mathbb{R}$ .  $\square$

#### 4.2.8 Team C

Figure 4.5 contains Chris' notes. The two students spend about three minutes reading the task, before they start to discuss one of the main points of the task (lines 5-8):

- 2       **Chris:** It's the Lipschitz property these functions have that prevents the  
               function from being vertical.
- 3       **Curt:** Yes, it is.
- 4       **Chris:** That the slope goes to infinity. The slope has an upper bound,  $K$ .
- 5       **Curt:** But that is, what the task is all about is really that you can replace  
 6       the distance, right, with an outer measure.
- 7       **Chris:** Yes,... measure that's also... that's also some kind of length.
- 8       **Curt:** Yes.  
               (30 sec. pause, Chris looks in their 'alternative' textbook, 'Real Analysis'  
 10       [Carothers, 2000])
- 11       **Chris:** It is well presented here. Real analysis, proposition 16.4, it says  
 12       – there is a section about the outer measure – that the measure is  
 13       really the length between two points, the length of an interval. (20  
 14       sec. pause) So if you .. then you could use this theorem, this example.
- 15       **Curt:** Wait a minute, I'll be right there. I am just writing down the defini-  
 16       tion of the outer measure.  
               (2 min. pause)
- 17       **Curt:** Proposition 16.4.
- 18       **Chris:** But it's just one-dimensional.
- 19       **Curt:** Yes. But that is not a problem here. We have a function from  $\mathbb{R}$  to  
 20        $\mathbb{R}$ .
- 21       **Chris:** What did you say?



**Figure 4.5** Chris' notes from task 2. His reproduction of the task formulation has been omitted. During the solution process Chris, at one point, illustrates  $f$  as a function from one real line to another (and not depicted in two-dimensional space).

24

**Curt:** I said, our function goes from  $\mathbb{R}$  to  $\mathbb{R}$ . So that is not a problem.

**Chris:** Oh yes. That is also one-dimensional.

**Curt:** Yes. (They laugh)

Chris starts to examine what properties the Lipschitz condition implies. At this point he does not mention that the Lipschitz condition implies continuity of  $f$ . Presumably by comparing the Lipschitz condition and the concluding inequality, Curt declares the main point of the task (lines 5-8). When he looks in the chapter concerning the outer Lebesgue measure in their 'alternative' textbook, [Carothers, 2000], he comes across a result stating that the outer Lebesgue measure of an interval is equal to the length of the interval. He thinks they might be able to apply this result. In the mean time, Curt starts to write down the definition of an outer Lebesgue measure from the textbook, but returns to discuss the proposition. From their dialogue (lines 18-25), it becomes clear that Chris, until now, has been unaware of the fact that the task situation is restricted to one-dimensional space. Thus, I suspect he notices the proposition because it exactly matches the identified main point of the task, and not because he compares the premises in the task with the proposition.

26

**Chris:** The function is thus satisfied when we have an interval [yes] but  $E$  isn't necessarily an interval ..

- 28 **Curt:** You could always turn it into one. You could say that you took the  
interval from  $x$  to  $y$ , when we know that it's the real numbers. No,  
30 yes, then you have to take, then you have to take the biggest, no, you  
don't know if that is right.
- 32 **Chris:** It could be the Cantor set.  
**Curt:** Yes, it could.
- 34 **Chris:** What happens then?  
**Curt:** Yes, that's right. Should  $\nu$  take the long way?
- 36 **Chris:** What if  $E$  was some kind of weird set? Then it would .. Or just a set  
that was divided into .. then it's just divided into some sub-intervals.
- 38 **Curt:** Yes, exactly. And then you would be able to do it with each one of  
them.
- 40 **Chris:** And then you could do it for each one.  
**Curt:** Yes.
- 42 **Chris:** And they have .. every sub-interval has a  $K$  and then you put  $K$   
equal to the maximum of all the  $K$ 's [yes] if there are finitely many  
44 intervals [yes]. And if there were infinitely many intervals, what then?  
Then there has to be ..
- 46 **Curt:** Yes, but. Well.  
(3 min. pause, they write something down)
- 48 **Curt:** Have you come any further?  
**Chris:** No, I can't say that I have.
- 50 **STO:** Do you feel that you just need to write it down? Or how do you feel?  
Are you thinking about it?
- 52 **Curt:** No, I am thinking about *if* we know that it's satisfied for an interval,  
how can we then be sure that  $E$  can be written by means of some  
54 intervals such that we can use what we know, that is this theorem  
from our textbook. Eh, yes.
- 56 (30 sec. pause)
- STO:** Could you, maybe, try – now I am interfering – to show that it's  
58 satisfied for an interval?
- Curt:** Yes, we could do that. We can start by saying that and assume that  
60 it's an interval [yes]. And then we just have to, then the distance from  
the endpoints is the same as the difference between the endpoints,  
62 right? That's..(He writes)
- (30 sec. pause)
- 64 **STO:** What did you write there, Chris?  
**Chris:** I wrote that, if I looked at the Lipschitz property and then I just  
66 assured myself that then it has to be .. that entails continuity of the  
function. Otherwise, you would destroy the Lipschitz [yes]. For me,  
68 it's all about rewriting this  $f(E)$  .. rewrite it to this form.
- STO:** Mmm.
- 70 **Curt:** Well, we also have the other.  
**Chris:** I just have to figure out how to rewrite this. Because when you have  
72 this, then you can get out from here and .. then you have a lot of nice  
properties, the outer Lebesgue measure. Linearity, for instance. What  
74 I am trying is to – you take a set and then – is to rewrite this so it  
looks like that.
- 76 **STO:** So you can use the property (the Lipschitz property)?  
**Chris:** Yes. That I can write it in a way that ends up being a good argument.

Chris (presumably) postulates that the concluding inequality (he calls it “*the function*”) is satisfied if  $E$  is an interval (line 26); Curt agrees and they immediately move on to discuss what to do if the set is not an interval (lines 28-46). Chris formulates what (they believe) they have to prove and a proof strategy; Curt seems to agree (lines 42-46). They think that they have to show that  $K$  exists such that the inequality is fulfilled and their strategy is to take the maximum value of all the  $K$ s for the different intervals in a covering of the set  $E$ .

I decide to interrupt and ask them to construct the proof in the special case where  $E$  is an interval. Chris writes down what he believes to be the Lebesgue measures of  $E$  and  $f(E)$  in this case (he introduces the intervals  $I_a^b$  and  $J_a^b$ , but it is not clear from his notes how  $I_a^b$  and  $J_a^b$  are related). Based on the transcript, it is difficult to determine whether the two students think they have proved the special case or if they get distracted during the attempt. When I ask Chris what he is writing, he explains that he is trying to convince himself that  $f$  must be continuous (lines 65-67). Afterwards he seems to have moved on to the general case (line 67). He is trying to rewrite the expression of the Lebesgue measure such that he can use the Lipschitz property directly.

Since the students do not indicate in any way that they find it difficult to prove the special case, I find it most likely that they actually believe they have proved it. I speculate that they have ‘constructed’ the following (insufficient) line of inferences ( $x$  and  $y$  are the endpoints of the interval  $I$ ):

$$\begin{aligned} |f(x) - f(y)| &\leq K |x - y| \Rightarrow \\ \ell(f(I)) &\leq K \ell(I) \Rightarrow \\ m^*(f(I)) &\leq K m^*(I), \end{aligned}$$

where the last implication follows from proposition 16.4. In the first implication it is (implicitly) assumed that  $|f(x) - f(y)| = \ell(f(I))$ , but this is only true for particular functions such as increasing or decreasing functions (as the first step in the proof shows, this implication demands more extensive justifications). Chris’ illustration of the function, see figure 4.5, might be the reason that they overlook this aspect. Had he illustrated the function in two-dimensional space, he would have been more likely to spot the mistake.

78 (25 sec. pause)

**Chris:** I guess you could write..you could write  $E$  as the union of intervals?

80 **Curt:** Eh. You still have to allow single points as intervals.

**Chris:** Then it’s just very short intervals.

82 **Curt:** Yes.

(70 sec. pause, they write and look in the professor’s notes)

84 **Curt:** Can’t we use this definition 3 in Michael’s notes? Note 4. The outer Lebesgue measure is defined as the infimum of the set of ..

86 **Chris:** Yes, it’s the same here (refer to the definition in the textbook).

**Curt:** Yes. But doesn’t that give .. when it’s infimum over that, then you could say, this interval .. there it’s more the definition of the Lebesgue

measure, which uses the interval. ... Then you just have to make a  
 90 sequence of intervals, where the union contains  $E$  and then we have to  
 say that the Lebesgue measure is infimum over the sum of the measures.  
 92 (20 sec. pause)  
**Chris:** Yes, we have to use that one.  
 94 **Curt:** Yes, I think so too. It is the easiest.  
 (2 min. pause, they erase something they had written down)  
 96 **Chris:** It's just not very operative, this definition.  
**Curt:** Isn't it? What if you insert? That is, you don't have to, you don't  
 98 have to specify exactly what covering it is, that works, you can just  
 keep on carrying this infimum. Or what? Do you have to take the  
 100 function eh, then we have to use what we know about the function?  
 (2 min. pause) Then it says .. You could write.  
 102 **Chris:** Well, the function is continuous, so for every sub-interval [yes] there  
 is an image interval and for that image interval, this property applies,  
 104 but maybe with a different  $K$  every time. And then we just have to  
 prove that there exists one.  
 106 **Curt:** No, not a new  $K$  each time or?  
**Chris:** Well, if it's already the biggest  $K$ , then there is no need.  
 108 **Curt:** If such exists, yes.  
**Chris:**  $K$ , it's satisfied for any of these sub-intervals [yes]. It's just the  
 110 matter of writing it down.

Without having completed the proof in the case where  $E$  is an interval, they return to discuss the situation where  $E$  is a general set (lines 79-82). They look up the definition of the outer Lebesgue measure and Curt sketches parts of a proof (lines 87-91 and lines 97-99). At least Chris does not find the definition operational (I speculate that he is confused about how to operate (algebraically) on the infimum of a number set). Curt suggests that they apply what they know about the function (line 100) and this encourages Chris to repeat the continuity property of  $f$  as well as to conclude that the image of a given interval in the domain is again an interval in the codomain (lines 102-103). Instead of using this information to make a (more) constructive illustration of the situation, however, they focus on the constant  $K$ . Again, it becomes clear that Chris has a faulty comprehension of what they are supposed to show. He believes that they have to show that there exists one constant,  $K$ , for which the inequality is satisfied (line 104), but Curt opposes the claim that different values of the constant exist for each interval in the covering of  $E$ . Chris quickly responds to Curt's critique by choosing the biggest constant of them all, just as he did previously. Curt does persist in maintaining his correct interpretation, but questions the possibility of choosing the biggest  $K$ .

At this point the students have spent about twenty-seven minutes talking and since they have not managed to prove the inequality in the case where  $E$  is an interval, I choose to stop the solution process and ask them to move on to another task.

As postulated in the beginning of this section, it is clear from the two students' dialogue that they possess a sufficient knowledge base. Among other things, they

know the Lipschitz condition and are able to derive properties from it. They are clearly familiar with the notion of the outer Lebesgue measure and they are able to connect it with the length of an interval. They notice that the inequality is satisfied for an interval (although they do not manage to prove it), and that the difficult part is the case where  $E$  is not an interval. They know that it is possible to choose a maximum value among finitely many numbers, but that this is not the case when there are infinitely many. Even though they seem to possess sufficient resources, they are neither able to prove the special case nor the general one.

The two students start by identifying what they see as the main point of the task, i.e. that the measure of a set is a generalised length of the set. This realisation does not provide any proof strategy so Chris begins by looking in the textbook that the two students have used to revise the measure and integration theory for the exam. In the section concerning the outer Lebesgue measure, he finds a result connecting the length of an interval and the outer Lebesgue measure. They believe they can use this in the proof, but still they have not clarified exactly what they have to prove.

They become occupied with other details. Sensing that they might run into difficulties if the set is not an interval, they try to fix this problem. Chris even makes an (incorrect) sketch of what this step in their future proof might look like. He also convinces himself that the function is continuous (another detail in the proof). At this point, they have still not formulated what they are supposed to prove.

They claim that the statement is satisfied if the set is an interval, but they are unable to provide the proof details of this special case. Chris attempts to rewrite  $f(E)$  into a form where he can use the Lipschitz condition, but he cannot figure out how to work out this detail even though Curt suggests that “*you can just keep on carrying this infimum*”. The attempts to solve the issue end with Chris’ argumentation that the image of an interval must again be an interval (since the function is continuous); again another detail in the proof. Once again, they reveal that they do not have an appropriate image of what they are supposed to prove.

### 4.3 The notions of structure, components and details

The overall outcome of the last thirty-one pages is that both structure and details are important notions when trying to understand the essence of students’ solving difficulties in relation to the way they have been taught. Admittedly, I have been a bit vague so far about how to define structure and details, but as used here structure includes the concepts, relations between concepts, the mathematical structure of analysis, and proof strategies, while details concerns notation, symbolism, procedures in proofs, and individual proof steps. In order to use the notion of structure and details to create a link between the teaching practice

and the solution processes, it is necessary to be more precise about the definition of structure and detail. With the aim of linking the solution processes and the teaching practice, it seems relevant to focus on the following two activities:

- Proof validation in class,
- Proof construction (including detection and formulation of mathematical claims when they are not explicit).

What does it take to carry out a proof validation or to be able to construct a mathematical proof? What do the students need to understand and in what ways are they supposed to understand it?

In order to answer these questions, I have found it fruitful to consider a theoretical construction composed of the *three* notions *structure*, *components*, and *details*. In general, a structure is composed of interrelated components, where the details of the components can vary in number and complexity. There is a dialectical relationship between structure, components and details. It is not possible to completely comprehend the structure if the details are unknown. Likewise, identifying something as a detail implies that it is a detail of a larger system.

#### 4.3.1 Proof validation

The following provides a proposed definition of structure, components and details concerning proof validation:

*The structure of a completed proof is a hierarchical network consisting of the main steps or components in the chosen proof strategy. The elements of the realization of the components are called the details of the proof.*

In a situation where students have to independently construct a proof of a mathematical claim, they have to decide on a proof strategy, construct the proof in a number of steps and, finally, provide the details of those steps. When the proof is already made, as is the case in a textbook, students have to identify the proof strategy used, the components the proof are made up of and the details of these components. In the proposed definition, the structure of a proof equals the hierarchy composed by the choice of strategy, the components and the details. The structure of a proof, and thus the components and the details, are not uniquely determined by either the statement or the chosen strategy. A statement can (sometimes) be proven using different strategies, and a chosen strategy can lead to different choices of components and details.

The main steps in a proof are often related in some way, but the details of one component may, besides having a relation to other details in the same component, also relate to details of other components in the structure. Relations between components and between the details of different components provide a network within the hierarchy.



As an illustrative example, I begin by analysing what it implies to validate a textbook proof with respect to structure, components, and details. Next, I use this theoretical construction to analyse the classroom dialogue when the professor validates the proof in class. Selected for the example is the theorem stating that the definite integral of a function of two variables is continuous:

**Theorem 11.4** *Let  $H = [a, b] \times [c, d]$  be a rectangle and suppose that  $f : H \rightarrow \mathbb{R}$  is continuous. If*

$$F(y) = \int_a^b f(x, y) dx,$$

*then  $F$  is continuous on  $[c, d]$ . [Wade, 2004, p. 325]*

The theorem contains three premises, and a conclusion:

$P_1 : H = [a, b] \times [c, d]$ ,

$P_2 : f : H \rightarrow \mathbb{R}$  is continuous,

$P_3 : F : [c, d] \rightarrow \mathbb{R}$  exists and is defined by  $F(y) = \int_a^b f(x, y) dx$ ,

$Q : F$  is continuous on  $[c, d]$ .

The textbook proof of the statement also justifies that  $F(y) = \int_a^b f(x, y) dx$  is well-defined, which is part of premise  $P_3$ . This means that the proof provided in the textbook actually proves the following statement:

*Let  $H = [a, b] \times [c, d]$  be a rectangle and suppose that  $f : H \rightarrow \mathbb{R}$  is continuous. Then*

$$F(y) = \int_a^b f(x, y) dx$$

*exists and is continuous on  $[c, d]$ .*

This statement contains two premises corresponding to  $P_1$  and  $P_2$ , and the third premise now becomes part of the conclusion:  $F = \int_a^b f(x, y) dx$  is well-defined and continuous. Based on the textbook proof, I identify the main components in the proof structure and explain the details of the components. The numbers in brackets refer to the different steps/components in the proof:

#### **Proof**

[1] For each  $y \in [c, d]$ ,  $f(\cdot, y)$  is continuous on  $[a, b]$ . Hence, by Theorem 5.10,  $F(y)$  exists for  $y \in [c, d]$ . [2] Fix  $y_0 \in [c, d]$  and let  $\epsilon > 0$ . [3] Since  $H$  is compact,  $f$  is uniformly continuous on  $H$ . Hence, choose  $\delta > 0$  such that  $\|(x, y) - (z, w)\| < \delta$  and  $(x, y), (z, w) \in H$  imply

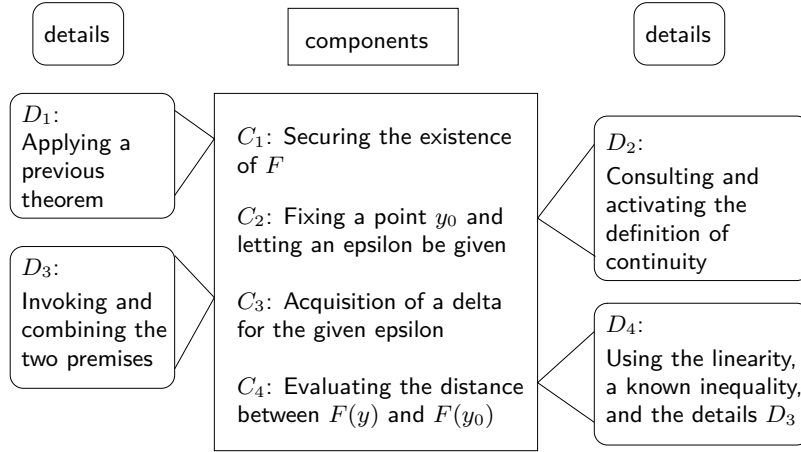
$$|f(x, y) - f(z, w)| < \frac{\epsilon}{b - a}.$$

[4] Since  $|y - y_0| = |(x, y) - (x, y_0)|$ , it follows that

$$|F(y) - F(y_0)| \leq \int_a^b |f(x, y) - f(x, y_0)| dx < \epsilon$$

for all  $y \in [c, d]$  that satisfy  $|y - y_0| < \delta$ . We conclude that  $F$  is continuous at the arbitrary point  $y_0$  and thus on  $[c, d]$ . [Wade, 2004, p. 325, numbers in brackets have been included]  $\square$

As the numbers in brackets indicate, I perceive the proof structure as being composed of four steps or components (this number is not uniquely determined), where the first is related to the existence of  $F$ , while the other three concern the application of the definition of continuity. Figure 4.6 illustrates the structure together with the details.



**Figure 4.6** The structure of the proof of theorem 11.4 is composed of the main steps or components that the chosen proof strategy leads to. The realisations of the components are the details of the proof. The components are denominated  $C_1$  to  $C_4$ , while  $D_1$  to  $D_4$  denominate the associated details.

The details of the proof describe what it takes to realise each of these four components. The first component includes the application of a previous theorem and ‘the validator’ has to confirm that the conditions for applying the theorem are satisfied. When this is done the result follows directly from the theorem.

The rest of the components describe what it takes to carry out the chosen proof strategy, which is to use the definition of continuity to show that  $F$  is continuous. The second step contains two conditions, a specified point and a given epsilon. The first condition ‘Fix  $y_0 \in [c, d]$ ’ is explained by the fact that in order to prove that a function is continuous on a given set, it is necessary to prove that the function is continuous at every point in that set. The second condition ‘let  $\epsilon > 0$ ’ follows directly from the formulation of the definition of continuity.

The third component contains the acquisition of a delta for the given epsilon. The need for acquiring a delta follows also from the definition of continuity. In order to find a delta, the two premises in the theorem need to be invoked and combined. The deduction that  $f$  is uniformly continuous (which is not presented in the textbook proof and actually draws on two previous proven theorems) provides a delta by which the distance between  $f(x, y)$  and  $f(z, w)$  is controlled.

The final component contains the evaluation of the distance between  $F(y)$  and  $F(y_0)$ , and the conclusion. The textbook omits some of the details of this component which the validator needs to infer:

$$\begin{aligned} |F(y) - F(y_0)| &= \left| \int_a^b f(x, y) dx - \int_a^b f(x, y_0) dx \right| \\ &= \left| \int_a^b (f(x, y) - f(x, y_0)) dx \right| \\ &\leq \int_a^b |f(x, y) - f(x, y_0)| dx, \end{aligned}$$

where the linear property of the integral and the (well-known) inequality  $|\int f| \leq \int |f|$  have been used. Now since,  $|y - y_0| = \|(x, y) - (x, y_0)\| < \delta$ , the following holds:

$$\int_a^b |f(x, y) - f(x, y_0)| dx < \int_a^b |\epsilon/(b-a)| dx = \epsilon,$$

where the fact that the integral of a constant equals the constant times the length of the interval. The result that  $|F(y) - F(y_0)| < \epsilon$  concludes the proof.

#### 4.3.2 Example of proof validation in class

Turning to the classroom presentation of the same proof, I focus on the emphasis the professor and the students put on the different components and details of the proof. The following episode begins after an hour of lecturing (including a break) about the differentiability of functions of several variables.

- Professor:** If we have a rectangle  $H$  (The professor draws a rectangle)  $a$  to  $b$  times  $c$  to  $d$ . You have a function small  $f$  from  $H$  to  $\mathbb{R}$ .  $f$  is continuous, so, eh, you could be tempted to say, but I want to have a function of one variable out of this, by averaging over the values, over the  $x$ -values at constant  $y$ . You could define a function capital  $F$  from  $[c, d]$  to  $\mathbb{R}$  by the integral  $F$  of  $y$  evaluated from  $a$  to  $b$  of  $f(x, y) dx$ ...you could well-imagine a situation where this could happen, where you could feel like doing that. So that means that every time you have a  $y$ -value in the interval from  $a$  to  $b$ , then I drive my harvester over here (He draws a horizontal line in the rectangle) and collects how much  $f$  there is along this...and then it is actually very plausible, that, eh, this function capital  $F$  has to be continuous at  $y$ , right?...And it is, actually.
- (Students laugh)

The professor introduces the situation in the theorem and motivates the construction of the integral. He refers to hypothetical situations where it would be relevant to construct a function which could produce the integral of  $f$  at a given  $y$ , but he does not provide any concrete examples. Nor does he explain why it is “*plausible*” that the defined function is continuous or discuss why it is even an interesting question to examine. The professor continues to talk about the content of the next theorem, theorem 11.5, and returns to the proof of theorem 11.4 after a couple of minutes.

By omitting the part about the existence of  $F$ , the professor shows the theorem as formulated in the textbook. He begins by concentrating the attention to the choice of a proof strategy, *“How do you show that something is continuous?”* (line 17). Alan replies by first referring to the high school definition of continuity

in terms of limits (line 19). This answer does not provide a proof strategy, so the professor indicates that he is looking for another answer, “*Yes, for instance*”. Alan tries to be more precise (line 21), and he now switches to the epsilon-delta definition of continuity. He is clearly not able to recite the definition and he reveals that the unusual notation ( $y_0$  is used instead of the usual  $a$ ) is a complicating factor for him. The professor gives them the hint to try to show that  $F$  is continuous at a point (line 24), and now he has actually provided the first condition in the second component.

Even though Alan is aware of his difficulties connected to the unusual notation, he is still unable to regard  $y_0$  as a constant when he attempts to give a definition of continuity now returning to the definition in terms of limits (line 27). The professor ‘translates’ what Alan is saying (line 29) and decides to write down his proposal (line 31). The professor’s aim is to get the students to recite the formal definition of continuity, but for some reason he does not want to ask them directly. Instead he tries to make them realise by themselves that the *meaning* of the notion of continuity lies in the formal definition, “*What does it mean?*” (line 34). He gives them the hint that they “*have to pull all that out of your pockets*”, and at this point Danny begins the ‘rhyme’. Betty continues it and the professor finishes it, correcting Betty’s words “*then there exists a delta*” to “*find delta*” along the way. The professor does not explicate the importance and consequence of his correction of Betty, and he does not make it clear that ‘finding a delta’ is an important and demanding step (component three) in the proof.

What becomes clear when the rest of the review of the proof has been presented, is that the professor at this point assumes that the students are able to see how the proof structure emerges once they have recited the formal definition of continuity. He does not repeat the second component, which was provided during the attempt to repeat the definition (“*let  $y_0$  be in an interval*” in line 25 and “*Let epsilon be bigger than zero*” in line 38), probably because he has already written it down on the blackboard. He continues with the details of the third component:

- 50       **Professor:** So we have to do something (non detectable). And what practically jumps out at you, is that  $H$  is closed and bounded.
- Student:** Compact.
- 52       **Professor:** Compact. (students laugh, the professor smiles) And what do we know about continuous functions on compact sets? ... actually, they
- 54       are a bit more fancy than continuous functions.
- Chris:** Their images?
- 56       **Professor:** They have a more fancy kind of continuity, haven’t they?
- Chris:** They are not only continuous, they are uniformly continuous.
- 58       **Professor:** Yes.
- Alan:** We are going to use our epsilons and deltas to put them to something?
- 60       **Professor:** F, yes.
- Alan:** Yes.

62     **Professor:** I can't remember the number of the theorem, but there was a  
 64         theorem in chapter 10.4, which says that if I have a continuous function  
 on a compact set, then it's uniformly continuous.

**Danny:** 10.52.

66     **Professor:** And it's called 10.52. So  $H$  is compact. That we could spin a yarn  
 over.  $H$  is closed and bounded with  $\mathbb{R}^2$  is Bolzano-Weierstrass and  
 68         then closed and bounded is the same as compact.  $H$  is compact entails  
 that according to 10.52, that  $f$  is uniformly continuous. This means  
 70         that there exists a delta bigger than zero such that for all pairs of  $(x, y)$ ,  
 $(x', y')$  in  $H$  with the difference between the  $(x, y)$  and  $(x', y')$  smaller  
 72         than delta, the difference between  $f(x, y)$  and  $f(x', y')$ , absolute value,  
 is smaller than, let's get some space, epsilon over  $b$  minus  $a$  (students  
 74         and professor laugh), but otherwise we have to divide by  $b$  minus  $a$   
 afterwards, so we might as well do it now.

The students follow along and participate in the professor's guided reasoning. The professor carefully provides all the details (also the ones the textbook omits), but he does not mention the component at all. He might believe that the component has already been emphasised sufficiently when he recited the definition, and that it is clear that he is trying to find a delta matching the given epsilon (that Danny provided). Alan presumably tries to interpret what is going on (line 59), but it is not easy to understand what he is alluding to. Chris also makes an attempt to interpret what they are doing:

76     **Chris:** So we expanded the starting point from a point to be all over?

**Professor:** (He steps back from the blackboard) I don't understand.

78     **Chris:** No. (He gives up)

**Danny:** You started by putting, by saying look at  $y_0$  and now you have, now  
 80         you have uniformity.

**Professor:** Yes, that's right. So in a way that gives, so you are saying  
 82         that this (meaning  $F$ ) is not only continuous, it will also be uniformly  
 continuous? That must follow from the proof, right? It does, because  
 84         it's a continuous function on a compact set, but it also follows from  
 the proof.

Chris supposedly tries to understand why it is necessary that  $f$  is uniformly continuous. As Danny explains what Chris is thinking about, and since he does not contradict Danny, it is reasonable to believe that Danny is providing a correct interpretation of Chris' question: they are trying to show that  $F$  is continuous, which implies that  $F$  should be continuous at every point in the set, so they look at a certain point  $y_0$ . In order to make the argumentation applicable to all points in the set, however, it is necessary that  $f$  is uniformly continuous and not just continuous. The professor misinterprets what Danny is saying, and believes that Chris and Danny are referring to the fact that  $F$  is uniformly continuous (which is true since the interval is compact). What Chris is actually trying to do is to understand the necessity of the details (that  $f$  is uniformly continuous) of the third component in relation to the second component and the statement they are

trying to prove. The professor responds by reformulating the conclusion  $Q$  to  $Q'$ :  $F$  is uniformly continuous. I can only speculate that this misunderstanding would not have been so likely to occur had the components and their place in the structure been explained more clearly.

- 86     **Professor:** Yes, well, what is, eh,  $f(y)$ ? We use of course this delta here,  
           now we have a candidate for delta.
- 88     **John:** Are you going to call it something special, or do you just call it delta?  
           (The professor does not hear the question)
- 90     **Professor:** Let  $y$  be in  $[c, d]$ , let the difference between  $y$  and  $y_0$  be smaller  
           than this delta.
- 92     **Chris:** What is the candidate for delta?
- Professor:** The delta that we found here (He points at the blackboard where  
 94                   delta is chosen to satisfy the epsilon in connection to uniformity of  $f$ ).  
                   We just have to calculate. (He writes ' $|F(y) - F(y_0)|$ ') )
- 96     **Alan:** It can't come from this theorem 10 or something? Our first, eh..
- Chris:** Where is the candidate?
- 98     **Student:** It is over there.
- John:** It is over there next to Michael.
- 100    **Professor:** We used that  $f$  was continuous and  $H$  compact to say that  $f$  is  
                   uniformly continuous. So when I have an epsilon then it gives me a  
 102                   delta and this is the delta that I have chosen.
- Chris:** And you do not know it more specifically?
- 104    **Professor:** No, we just need some delta, right? (He shrugs his shoulders).

Quickly, the professor mentions the third component (lines 86-87), and this initiates confusion. The class has just spent some time finding the delta, but even though Chris has participated actively in the process he does not realise they have found a candidate for delta. He has been able to follow the details of the third component, but he has not understood the relation between the details and the component. The professor repeats the details and connects them to the component (lines 100-102), and Chris reveals that his confusion also has to do with the fact that the delta found is not specific (line 103), which seems to be the same reason why John asked if the professor intended to rename delta (line 88).

- 106    **Professor:** So there I have the difference between  $F(y)$  and  $F(y_0)$ , I have to  
           have some kind of formula to do any calculations, right? Otherwise it's  
           a bit difficult (He speaks out aloud while he writes ' $|\int_a^b f(x, y) dx -$   
 108                    $\int_a^b f(x, y_0) dx|$ '). ... can we do something about this? (10 sec. pause)
- 110    **Danny:** You can start by, and, that is, the integral is linear, so you may  
           write the integral of the differences between the functions.
- 112    **Professor:** So we just use the linearity of the integral to say that the integral  
           from  $a$  to  $b$  of  $f(x, y)$  minus  $f(x, y_0)$   $dx$ . That's just linearity, right?  
           (He speaks out aloud while he writes ' $|\int_a^b (f(x, y) - f(x, y_0)) dx|$ ') That  
 114                   was just linearity. We have done nothing, eh, we can (corrects ' $<$ ' to  
                   ' $=$ ') and then we can put the absolute value signs inside.

- 116 **Danny:** You also have to put the other one in. (The professor missed one of  
the absolute value lines)
- 118 **Professor:** Thanks. Can you see what we can use? Because what is the  
difference? How far is  $(x, y)$  from  $(x, y_0)$ ?
- 120 **John:** The  $x$ -coordinates?
- Professor:** On the whole?
- 122 (15 sec. pause)
- Professor:** They have the same first coordinates, right? It is actually the  
124 difference between the second coordinates. And this is delta at the  
most. So the difference, so the distance, right, to the argument of  $f$   
126 here and the argument of  $f$  here, it's smaller than delta. And then  
it was, if we had a situation like that then the absolute value would  
128 be smaller than epsilon over  $b$  minus  $a$ . (He looks at the students and  
returns to the integral of the numerical value). This must then be  
130 smaller than or equal to the integral from  $a$  to  $b$  of epsilon divided by  
 $b$  minus  $a$ , and that I can do in my head, it's epsilon.
- 132 **Humfrey:** The last inequality, is that strictly smaller than?
- Professor:** It is actually strictly smaller than. (He corrects it)
- 134 (10 sec. pause)
- Professor:** So the delta we have caught from uniform continuity gives this.  
136 (He points at ' $|F(y) - F(y_0)| < \epsilon$ ').
- (45 sec. pause, the professor looks at the students and then in the textbook)
- 138 **Betty:** But we can write uniformly continuous? (The professor looks un-  
comprehendingly) We conclude that  $F$  is uniformly continuous?
- 140 **Professor:** Yes, you can say that. Or it follows from our proof by using a  
theorem for one variable functions. So if  $F$  is continuous, no if small  $f$   
142 is continuous as a function of two variables, then capital  $F$  is continuous  
as a function of one variable  $y$ .

The professor continues with the details of the fourth component. A point  $y_0$  in the set has been chosen, an epsilon has been given (the second component), a delta independently of  $y_0$  has been found (the third component), and now it is time to evaluate the difference between the images. Again the professor does not refer to the component and the structure (he could have mentioned the component by saying "*To conclude that  $F$  is continuous we now need to evaluate the difference between the images*"), but goes directly to the details (line 108). Accompanied by Danny, the professor evaluates the difference. He uses the conclusion from the third step ( $|f(x, y) - f(z, w)| < \epsilon/(b-a)$ ) and gets to the endpoint, that  $|F(y) - F(y_0)| < \epsilon$ . Betty wants to be sure that he meant what he said earlier (his answer to what he thought was Chris' question intended) and asks if she can write that  $F$  is also uniformly continuous (lines 138-139). She thus tries to confirm that the details have proved the sharpened conclusion,  $Q'$ :  $F$  is uniformly continuous. At first the professor confirms this ("*Yes, you can say that*"), but his subsequent explanation does actually not justify that  $F$  is uniformly continuous.

Overall, what happens in the dialogue? First, the professor attempts to clarify the structure of the proof by drawing the students' attention to the epsilon-delta definition of continuity. During the attempt, he 'takes care' of the second



component. It seems that the professor is now under the impression that the proof structure has been made clear to the students, so he continues with the details of the third component. When a student tries to gain understanding of the relation between a detail of the third component and the second component, the professor misinterprets the question and restates the conclusion ( $Q \rightarrow Q'$ ). The professor then returns to the third component, where the students now seem confused about the relation between the details and the component. The details of the fourth component follow without any mention of the fourth component.

The analysis of the proof validation process shows that the professor addresses the connection between the statement and the chosen proof strategy, but he never explicitly says that they are going to use the definition of continuity. To a large extent he takes for granted that the students are able to see how the proof structure follows from the chosen proof strategy. The proof structure and the individual components are not explicitly pointed out. Since the proof structure is taken for granted the details are emphasised at the expense of the components and the structure in the professor's review of the proof. When components get mentioned it is after the details of that component have been explained. The dialogue also shows an example of miscommunication where a student tries to comprehend the complicated hierarchy of details within the proof structure, but is met by an insignificant reformulation of the conclusion. The reformulation prompts another students' need for clarification. All this presumably leads to the observed confusion in the students, and it is likely that the proof review has not made them realise the connection between the statement and the proof strategy nor made them able to see how the proof structure emerges from the chosen proof strategy. Even though they were able to follow the professor's explanations of some of the details, the details lost their meaning when the students could not see the relation to the components and the structure.

#### 4.3.3 Proof construction

When the aim is to use the theoretical construction as a tool for analysing proof construction situations, it is necessary to infer a proof of the claim at issue which can be subjected to a structure-component-detail analysis.

A unique presentation of a proof of a given claim does not exist, and many claims can be proved by different proofs. When producing a proof to be subjected to analysis, a decisive choice has been made which will affect the analysis of the solution process. In this respect, the subsequent analysis of students' proof processes is not unique. The proof has been made with an eye to features of the actual solving process, since it would be absurd if in the analysis of team C's solution process of task 1, for instance, I used the analysis of the proof based on the metric definition of continuity.

Nearly the same definition of structure, components and details that was used in the analysis of a proof validation process can apply here:

*The structure of a proposed proof is a hierarchical network consisting of the main steps or components in the chosen proof strategy. The elements of the realisation of the components are called the details of the proof.*

Although only one word separates the two definitions (the fifth word, *completed* versus *proposed*), there are several differences between a proof validation process and a proof construction process. It is reasonable to assume that a successful solution process demands the ability to validate the correctness of the inferences proposed and that the line of argumentation proves the claim. Contrarily, it does not seem necessary to possess the ability to construct proofs in order to validate an already completed proof.

To be able to construct a proof, the solver has to come up with a ‘good idea’ or identify the key idea [Raman, 2003]. This has already been done in a completed proof, and the job of the validator is (simply) to uncover it.

To illustrate how the definition of structure, components and details can be used to analyse a task solution process, I return to task 1 and then to task 2.

#### 4.3.3.1 Task 1

The task formulation is listed on page 94 and in appendix D.2. The task contains three premises:

- $P_1$**  :  $(M, \sigma_{discrete})$  is the discrete metric space, i.e.  $\sigma_{discrete}(x, y) = 0$ , if  $x = y$  and  $\sigma_{discrete}(x, y) = 1$ , if  $x \neq y$ .
- $P_2$**  :  $(M, \sigma_d)$  is an arbitrary metric space, e.g.  $\sigma_d$  is an arbitrary metric.
- $P_3$**  : The identity function  $i : (M, \sigma_{discrete}) \rightarrow (M, \sigma_d)$  is defined as  $i(x) = x$ ,  $x \in M$ .

The conclusion is not provided in the task, so it is not a traditional proof task. In fact, the task does not even ask for a proof, directly, but assumes that the solver has taken part in a teaching practice where the sociomathematical norm of proper forms of argumentation has been established such that the solver knows that he or she is expected to construct a proof. I use *inquiry* ( $I$ ) to denote what the solver is asked to examine. After the task has been solved it is possible and expected to transform the inquiry (or inquiries) into a conclusion(s).

**$I_1$**  : Determine if  $i$  is continuous.

**$I_2$**  : Determine if  $i$  is uniformly continuous.

As already described, determining whether the function  $i$  is continuous or not can be approached in two different ways by using the definition of continuity in a metric space or by using the theorem characterising continuity in terms of open sets (this result becomes a definition in a topological space setting). The question of uniform continuity can only be solved using the metric definition. Since I use the solution processes of team A (only the part presented above, starting on page

106) and team D as examples (presented at page 95), the proof will be based on the metric definition.

#### 4.3.3.2 Proof of statements in task 1

The following description of a proof is meant to illustrate a proving process and is not an attempt to make a proof as it would appear in a textbook.

If an obvious quick way to prove a mathematical property does not present itself, it is always possible to use the definition of that property. In this case it is the definition of continuity in a metric space setting that is relevant. This means the solver has to translate the inquiry into the formal definition of continuity on metric spaces (definition 10.27 in appendix C.1). Using this definition a proof could have the following form:

##### Proof

[1] To be continuous means that for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that expression (C.2) in definition 10.27 holds. So if  $i$  were to be continuous the following should hold: For  $a \in M$ , let  $\epsilon > 0$ , we can choose  $\delta > 0$  such that:

$$\sigma_{discrete}(x, a) < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad \sigma_d(i(x), i(a)) < \epsilon. \quad (4.1)$$

[2] Since  $\sigma_{discrete}(x, a)$  can only obtain two values, zero and one, it is possible to ‘force’ the metric function to take the value zero by choosing any  $\delta$  smaller than or equal to one. If  $\sigma_{discrete}(x, a) < \delta$  this means  $\sigma_{discrete}(x, a) = 0$ , so  $x = a$ . [3] Then it is time to evaluate the difference between the images:

$$\sigma_d(i(x), i(a)) = \sigma_d(x, a) = 0 < \epsilon,$$

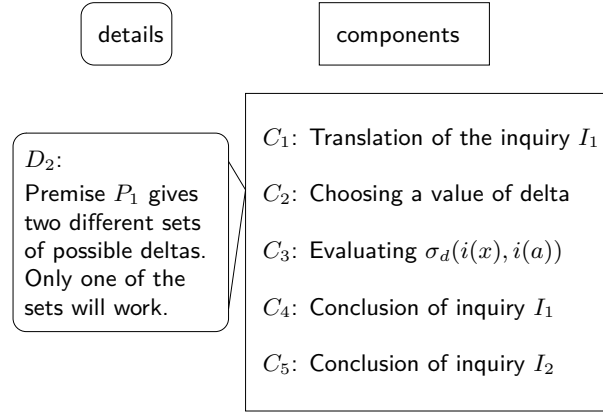
since  $i(x) = x$ ,  $i(a) = a$ ,  $x = a$ , and  $\epsilon > 0$ . [4] Let  $a \in M$ , let  $\epsilon > 0$ , choose  $\delta = 1/2$ . Then  $\sigma_{discrete}(x, a) < 1/2 \Rightarrow x = a$  which entails that:

$$\sigma_{discrete}(x, a) < \delta \quad \Rightarrow \quad \sigma_d(i(x), i(a)) < \epsilon,$$

according to [3]. Then according to [1],  $i$  is continuous. [5] Since the choice of  $\delta$  does not depend on the value of  $x$ ,  $i$  is also uniformly continuous.  $\square$

The components of the proof correspond to the numbered steps in the proof. Figure 4.7 illustrates the structure together with some of the details.

The details of the proof, referred to as  $D_1$  to  $D_5$ , respectively, define the realisations of the components. In the first step, the solver has to realise or remember that there exists a formal definition of continuity in a metric space (which is more general than the definition of continuity on  $\mathbb{R}$  with the standard metric). Even though the solver remembers this, the translation is not straightforward because the task is an inquiry task. The translation of the first component can be complicated by the fact that the task introduces several different concepts and notations that the solver has to figure out. Two metrics are introduced. One of them is a specific metric, which has been introduced in the textbook. The other



**Figure 4.7** The structure of the proof of the inquiries in task 1 is composed of the main steps or components that the chosen proof strategy leads to. The realisations of the components are the details of the proof. The components are denominated by  $C_1$  to  $C_5$ . The details,  $D_2$ , of the second component are provided as an example, whereas the other details are omitted.

one is an arbitrary metric and the only information the solver would be able to know is that this metric fulfils the three conditions given in the definition of a metric space. Next, a specific function is introduced. Although it is a simple function described in the task, the solver has to realise what the function does. To translate the inquiry, i.e. to provide the details,  $D_1$ , the solver has to realize the relationship between the two metrics, the identity function and the set  $M$ .

The second component entails the solver determining what it is he or she wants to try to prove, i.e. whether  $i$  is continuous or not continuous. Students will most likely try to show that  $i$  is continuous, since many students find it difficult to negate expressions that include multiple quantifiers (references are listed on page 19 and page 53). Hence, the aim of the second component is to choose a delta such that the definition of continuity is satisfied. The details,  $D_2$ , rest on the definition of the discrete metric. Since this metric can only obtain one of two values, there are only two fundamentally different sets of values for delta, either  $\delta \leq 1$  or  $\delta > 1$ . If the latter is chosen, the distance between the images is not controlled in any way. If  $\delta \leq 1$ , and  $\delta_{discrete} < \delta$ , then  $x$  is equal to  $y$ , and since  $i(x) = x$  and  $i(y) = y$ , premise  $P_3$  and the property of an arbitrary metric, premise  $P_2$ , ensures that the distance between the images is controlled.

The third component contains the evaluation of the difference between the images using  $P_2$ ,  $P_3$ , and  $D_2$ . The fourth component requires students to realise that the definition of continuity is satisfied enabling the conclusion that the function is continuous. In this case, the details of component one,  $D_1$ , need to be revisited. To justify that the function is also uniformly continuous, the fifth component, the students have to invoke the definition of uniform continuity (def-

inition 10.51 in appendix C.1), and realise, by going through the details,  $D_2$  and  $D_3$ , that the argumentation does not require point  $a$  to be chosen before delta.

#### 4.3.3.3 Task 2

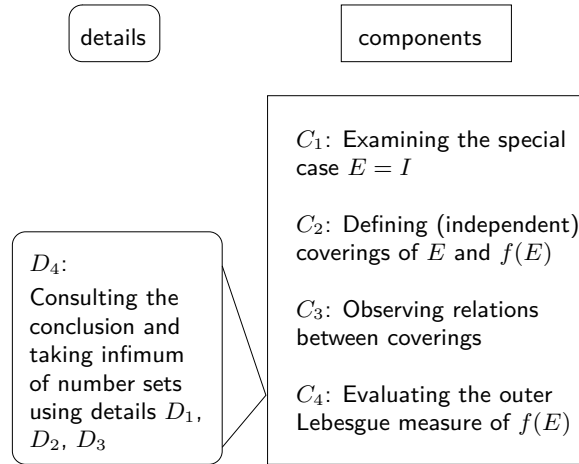
The task formulation is listed on page 118 and in appendix D.2, followed by the proof. There are two premises and one conclusion in the task:

$P_1$  :  $f$  is a function from  $\mathbb{R}$  to  $\mathbb{R}$ .

$P_2$  :  $f$  is  $K$ -Lipschitz.

$Q$  :  $m^*(f(E)) \leq Km^*(E)$ , for all  $E \subseteq \mathbb{R}$ .

Contrary to task 1, task 2 is an (ordinary) proof task, where the conclusion is provided in the task. Figure 4.8 shows the structure of the proof presented.



**Figure 4.8** The structure of the proof of the claim in task 2 is composed of the main steps or components that the chosen proof strategy leads to. The realisations of the components are the details of the proof. The components are denominated by  $C_1$  to  $C_4$ . The details,  $D_4$ , of the fourth component are provided as an example, whereas the other details are omitted.

As before, the details of the proof provide the realisations of the components, and are labelled  $D_1$ ,  $D_2$ ,  $D_3$ , and  $D_4$ . The details of the first component are extensive and seem technical, but they are really not that complex. A solver has to compare the definition of the outer Lebesgue measure to the definition of the Lipschitz condition in order to realise the usefulness of examining the special case where  $E$  is an interval. The student has to realise that it is not possible to formulate the Lipschitz condition in terms of the length of intervals (rather than the distances between endpoints) without providing justification, since the interval  $f(I)$  is not necessarily defined by the images of the endpoints of  $I$ . To

provide the justification, the solver has to invoke the Extreme Value Theorem, and know (or show) that the Lipschitz condition entails uniform continuity.

To realise the second component, the conclusion Q and the definition of the outer Lebesgue measure must be invoked. The conclusion explains why a set,  $E$ , is defined; combined with the definition, this explains why coverings of  $E$  and  $f(E)$  are needed. Because  $f$  is a one-variable function, the coverings consist of intervals (and not squares or cubes etc.). Furthermore, since the definition of the outer Lebesgue measure involves that infimum is taken over all coverings, it causes no loss of generality to look at coverings of disjoint intervals only (this detail actually involves proposition 16.1).

The third component provides the link between the two independent coverings through a third covering, namely the image of the covering of  $E$ . The solver has to invoke a consequence of premise  $P_2$ , i.e. that  $f$  is uniformly continuous, to realise that the image of the intervals,  $f(I_n)$ , is also a sequence of intervals and that their union provides a covering of  $f(E)$ .

To realise the fourth component, the conclusion must be invoked and combined with the results of the previous three components. Based on the conclusion, looking at and evaluating  $m^*(f(E))$  is of course reasonable. The definition of the outer Lebesgue measure and the details  $D_2$  provide the first equality sign. The following inequality sign is explained by the details of the third component,  $D_3$ . The second inequality sign follows from  $D_1$ , and the subsequent equality sign only demands the realisation that the summation and the infimum are not influenced by the constant,  $K$ . The final equality sign only demands an inspection of the definition of the outer Lebesgue measure. During the evaluation of the outer Lebesgue measure of  $f(E)$ , the solver must know how to evaluate expressions containing an infimum of number sets. This aspect entails that the details of  $C_4$  are more than just a combination of the details from the other components in the structure.

#### 4.3.4 Examples of proof constructions

The purpose of this analysis is to look for signs of structure, components and details in the solution processes to find out what the students focus on. It is to be expected that a solution process will begin with a clarification of the concepts and the notation used in a task. This stage, if it occurs, is included in the first step/component of the solution process.

The attempts of teams A and D to solve task 1, and team C's attempt to solve task 2, are used as examples. These three processes have been chosen because they represent the different types of solving processes observed (students with insufficient resources, students with sufficient resources who reach an answer, and students with sufficient resources who do not reach an answer).

#### 4.3.4.1 Revisiting team A and task 1

In the following,  $C_x$  and  $D_x$  refer to component  $x$  and detail  $x$ , respectively, described earlier (see figure 4.7 and the description of the components and details starting on page 136).

The students begin with a discussion about the two metrics (page 106, lines 1-17), which can be characterised as the details of the first component. The students go directly to the details without mentioning or referring to the component, so the details are detached from the component, which in turn implies a detachment from the structure.

The students are not precise when it comes to realising the details,  $D_1$ . Alan manages to write down a non-operative definition of the discrete metric (non-operative since it is unclear under what conditions the discrete metric becomes zero or one). Although Anna explains that an arbitrary metric is just a random metric, neither of them clarifies that a random metric is one which satisfies the definition. Anna proceeds to the inquiry stated in the task (page 107, line 28). This could have been a good opportunity to discuss component  $C_1$ , but Alan is still focused on the details,  $D_1$ . His suggestion to draw the situation moves their attention away from component  $C_1$  (page 107, line 30). Anna later makes another attempt to talk about component  $C_1$  (page 110, line 81), but now the (misinterpreted) details of the first components disturb the attempt. Alan interrupts with speculations about the conclusion of the inquiry. He doubts that continuity is possible when the situation is as he believes it to be. Now it is Anna's turn to disregard an opportunity to devote attention to the first component and she returns to speculations about the details (page 110, line 87). Alan then suggests that they should check the definition of continuity, which can be seen as an attempt to focus on  $C_1$ .

Then there is a long passage where the students alternate between talking about the conclusion of inquiry I1 (page 110, lines 103-110, and page 112, line 171) and the details of the first component (page 111, lines 123-134, page 112, lines 152-161), touching at times on the first component (page 110, lines 112-114, and page 112, line 165).

In this example two students are trying fairly hard to make sense of the task. They try different approaches, but without success. One characteristic of their process is that they spend a lot of time concerned with the details of the components, but without being clear about the actual components. This leads to conflicting feelings in the students (page 108, lines 76-77). They believe they are on the right track but at the same time they sense that something is missing even though all the necessary mathematical information is present (page 108, lines 67-70 and page 108, lines 77-79). Unable to see how all the details fit together in the structure, they cannot construct a proof.

They never manage to clarify component  $C_1$ , but they spend a lot of time discussing the details of this component. The fact that the task is not a traditional proof task, but an inquiry task, may have an effect. The interpretation of the

situation in the task is mixed up with guesses about the answer to the inquiry, which is the details of  $C_4$ . Since it is impossible to study what effect changing the task to a proof task might have had on these students, I can only speculate that it would have been easier for them to interpret the situation if they had known that the function was continuous.

The purpose of analysing a solution process like the one above is to find answers to why the two students do not succeed in solving the task. Doing an analysis from the perspective of structure, components and details shows that they pay significant attention to details and less or no attention to structure. The picture, however, is overshadowed because of the students' obvious lack of the necessary mathematical resources.

#### 4.3.4.2 Revisiting team D and task 1

This example shows what happens (from a structure, components, and details perspective) when a solution attempt succeeds. In team D, Danny has a totally different approach than the one Alan and Anna use, while Dylan acts more like Alan and Anna (page 95, line 2). When Danny asks Dylan whether he understands the task, Dylan refers to a detail in the task, namely the definition of the discrete metric, which is a detail connected to the first component. Danny explicates what he means by understanding the task (page 95, lines 7-11): it is to uncover the main point in the task. Danny is thus able to use the information provided in the task to decipher what the main point is.

The identification of the main point seems to provide him with an idea of how to construct the proof. For him, the structure of the proof emerges from this identification, and the construction of the proof becomes a matter of providing the details, because the structure is clear. The details of the first component take some time to fall into place. At first, Danny writes down an erroneous definition of continuity (page 95, lines 11-24), but it is clear that he is not confused about this component and how it relates to the proof structure. He continues with component  $C_3$  (page 96, line 29), where he evaluates the images. Then he suddenly 'sees' the solution (page 96, line 32), gets around the details of component  $C_2$  (page 96, line 33) and finishes with component  $C_4$  and the associated details (page 96, line 34). He remarks that the argumentation provided is independent of the location of the quantifier, so the function is also uniformly continuous. Danny is (implicitly) guided by the components and is able to provide the necessary details when needed.

#### 4.3.4.3 Revisiting team C and task 2

In the following,  $C_x$  and  $D_x$  refer to component  $x$  and detail  $x$ , respectively, described earlier (see figure 4.8, and the description of the components and details starting on page 138).

In the attempt to solve task 2, the two students in team C immediately identify the main point of the task. Contrary to team D, however, the identification of



a main point does not provide a constructive proof strategy. The fact that Curt claims that the purpose of the task is that the distance between two points can be replaced by an outer measure, makes Chris notice proposition 16.4 in the textbook. This result, and not the definition of the outer Lebesgue measure, becomes the guiding factor in their solution process. They think of the result as a ‘short cut’, but it has some non-constructive consequences for their process, which I shall return to.

Through what seems to be a superficial inspection, Chris concludes that the inequality is satisfied for an interval (page 120, line 26), and this actually gives them the result of the first component, but they cannot provide the right details. They proceed to  $C_2$  and describe parts of the details,  $D_2$  (only regarding the covering of  $E$ , page 121, lines 28-37). They conclude that they hereafter can use the inequality for each interval (this corresponds to  $D_4$ ) (page 121, lines 38-46).

Curt formulates their difficulties very clearly (page 121, lines 52-55): they do not know how to write the set,  $E$ , as composed of a set of intervals, and this can be translated into difficulties in providing the details of  $C_2$ . Then Chris, presumably without knowing what to do with the result, examines whether the function  $f$  is continuous. As a result, he addresses the details,  $D_1$  and  $D_2$ , but without being clear about which component they relate to or how they fit into the structure (page 121, lines 65-67).

The definition of the outer Lebesgue measure enters the picture again (page 122, line 84), but this time both Chris and Curt recognise the usefulness of it, maybe because they have been discussing  $D_2$ . Curt now formulates parts of  $C_2$  and  $C_4$  (page 123, lines 89-91) and parts of the details  $D_4$  (page 123, lines 97-100). He proposes that they use the properties of the function, which makes Chris address the details,  $D_3$  (page 123, line 102), but without knowing the component to which they belong. He still hangs on to his erroneous idea about what they are supposed to prove. Although Curt makes some objections, Chris only alters minor steps in his argumentation (*“Well, if it’s already the biggest  $K$ , then there is no need”*).

The two students clearly focus on the details of the components without being clear about how exactly the details relate to the components. They identify some of the components, for instance, the first component and parts of the second component, and sketch very loosely the details of the fourth component. They are unable to provide the details of the first component and they spend a lot of time on the details of the second component. They do not know how to provide the details,  $D_2$ , because they do not have a clear idea of how to proceed afterwards. When Curt spots the formulation of a covering in the definition of the outer Lebesgue measure, it seems that he now realises how the second component relates to the proof structure. This enables him to see how they can express  $E$  in terms of intervals.

The process and the analysis show that the students experience difficulties providing the details of the components and that they are unable to see the

structure of the proof clearly. They are focused on the details, but they do not see which components the details are details of. They cannot complete the details because they do not see the structure and they cannot construct the structure because they cannot complete the details. This is an example of the dialectic relation between the structure of a proof and its details.

#### 4.3.5 Formulating a hypothesis

Using the framework of structure, components and details makes it possible to characterise the interplay between a structural and detailed focus in the solution processes. The examples provided illustrate that team A had a very detail-oriented focus and that the details of the different components were mixed together. Team D, on the other hand, concentrated on the details of one component at a time, because the structure of the proof became clear to them early in the process. In team C, the students touched upon many of the details in the proof, and some of the components, but they were not able to provide the details, because the structure of the proof remained unclear to them and vice versa.

The analysis of a proof validation situation in class indicated that the proof strategy and the resulting proof structure had not been made clear to the students. Since the structure was not explicit, the students found it difficult to comprehend and ascribe meaning to the details. Assuming that the structure was clear to the students, the professor shifted between a structural and detailed focus without making the shifts explicit.

Comparing the conclusions from the solving sessions and the teaching practice provides empirically founded motivation for formulating the following hypothesis:

*The lack of clarity about what structure and details are in the validation process of a textbook proof in class can contribute to an explanation of the students' difficulties constructing new proofs on their own.*

In the following chapters, the data from the main study will be used to probe the research questions further and to examine this hypothesis.



## 5 Characterisation of teaching practice

“The nature of classroom mathematics teaching significantly affects the nature and level of students’ learning.” (James Hiebert and Douglas A. Grouws)

The overall aim of this doctoral project is to examine the effects of (a certain) mathematics teaching practice on student learning where learning outcome is measured through student solution processes. It is a fundamental question in mathematics education research how teaching affects learning, and it is likely – or at least not rejected by any empirical findings or theoretical arguments – that different teaching methods or styles lead to different learning outcome [Hiebert & Grouws, 2007, p. 374]. When analysing the learning outcome, it is necessary to ask what the purpose or learning goal of the teaching was. Robitaille distinguishes between three kinds of learning goals or curriculum goals:

The goals of the mathematics curriculum may be considered at any of three levels: intended, implemented and realized. The ‘intended’ goals are those promulgated by curriculum developers. They are set out in the teachers’ editions of textbooks, and are listed in curriculum guides. The ‘implemented’ goals are the goals of the curriculum as they are understood and implemented by teachers in their classrooms. Finally, the ‘realized’ goals are those attained by the students ... [Robitaille, 1981, p. 149]

Thus, the intended goals are goals that are expressed (in writing or verbally) either by government, the mathematics department or the professor in charge of the course, while the implemented goals are the goals that the particular teaching practice that is executed actually pursues. What the students might gain from participating in a particular course constitutes the third level, the realised goals.

A mathematics course at university level is governed by a set of more or less official descriptions. First, the course has a name which, for instance, refers to the mathematical content domain or curriculum, e.g. algebra or complex analysis, or indicates that the course concerns mathematical competencies across mathematical domains, e.g. mathematical modelling (mainly the modelling competency, as defined by Niss et al. [2002]) or introduction-to-proof (mainly the reasoning

competency, as defined by Niss et al. [2002]). At the university where I conducted the pilot study and the main study, the department determines the name of the course, the number of lessons per week and the duration of the course. The department also provides a rough sketch of the mathematical content, but the professor in charge of the course has rather free reign regarding choice of textbook, curriculum, and teaching style. For instance, while teaching the course observed in the pilot study, the professor chose to introduce the Lebesgue integral instead of the Riemann integral for functions of several variables. He did not need to consult the department on that decision.

The aims of the course can be found by reading department descriptions and the professor's description of the course, and by interviewing the professor. These goals can be explored before the course is held.

The implemented goals, which are the ones the actual teaching pursues, cannot be determined beforehand. This means that even though a department or a professor has formulated a set of learning goals, it might not be the same as the goals a particular teaching practice actually pursues. The learning outcome of the students, i.e. the realised goals, might again not be the same as the goals implemented. In fact, it is one of the dilemmas of teaching that the goals implemented by the teachers will in practice never be the same as the goals realised by the students.

This chapter concerns the first two goals, the intended and the implemented goals, while the next chapter concerns the third level, the realised goals. In this chapter, the professor's intended learning goals as expressed in interviews and the course (in the main study) description on the home page and during the first class will be presented first.

After this, an analysis of the learning goals implemented is presented. Four different analysis approaches have been chosen which focus on different levels of the teaching practice. First, an analysis based on the observation template developed is presented. The observation template is used to identify ten main elements characterising the teaching practice and to provide a global characterisation. A protocol analysis tool developed by Schoenfeld [1985] is adopted to provide time-line representations of different lessons. The focus then shifts to the actual interaction between the professor and the students, where the analysis more specifically concerns the establishment of social and sociomathematical norms in the classroom. While mathematical proof plays an important role at university level and in mathematical analysis, it seems relevant to focus on what kind of proof conceptions or proof schemes the teaching practice facilitates. This then provides the third analytic perspective.

In order to compare and contrast the analyses and the findings, the teaching practice observed in the supplementary study is invoked and subjected to analysis as well.

This chapter (except section 5.10) concerns and uses data material primarily from the main study. The professor interviews from the pilot study will be in-

cluded to elaborate on the professor's intentions, opinions, and beliefs, but this will be evident from the context.

## 5.1 Written learning goals

The course in the main study is titled 'Mathematical analysis and fundamental theory'<sup>1</sup>. The description of the course provided by the study board states that the course "*is concerned with the more subtle properties of the real numbers and especially with infinitesimal calculus, including concepts such as continuity, differentiability and the integrability of functions of one ... variable.*" [Larsen, 2005, p. 11, my translation].

The professor in charge of the course has to make additional specific decisions regarding content and teaching methods. In a written description of the course, the professor presents the mathematical content of the course and a list of general goals. The course primarily concerns

one-dimensional real analysis. Emphasis is on the conceptual foundation, on the construction of a coherent theory, and on detailed arguments for results, of which some might be well-known. Number sequences, convergence, infinite series [of real numbers] and convergence theorems. The Weierstrassian analysis. The concept of function and a systematic introduction to continuity and differentiability. Sequences of functions and power series. The Mean Value Theorem. Systematic introduction of the Riemann integral in one real variable. Proof types. Complex numbers. Applications in science. (From the course home page, my translation)

The general goals of the course are formulated within the framework of mathematical competence as defined in [Niss et al., 2002]:

The goal is for students

- to work with and develop their *mathematical representation competency* and *symbols and formalism competency*, and through this become fluent in the symbolism and formalism of mathematical analysis,
- work with and develop their *mathematical thinking competency*, and through this become confident with the mathematical concepts introduced, including their range and mutual relations,
- work with and develop their *mathematical reasoning* and *communication competencies*, and through this learn to read, analyse, understand and construct mathematical proof orally and in writing within the conceptual frame of [mathematical] analysis,
- work with and develop their *mathematical problem solving competency*, and through this gain confidence concerning questions and issues where mathematical analysis enters in a substantial way.

(From the course home page, my translation and italicisation)

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<sup>1</sup> In Danish: Matematisk analyse og grundlæggende teori.

After the specific content related goals, the professor lists two “*specific minimum goals*”:

The student should be able to demonstrate confidentiality with

- the Weierstrassian analysis,
- the close relationship between, on the one hand, concepts and results of [mathematical] analysis, and, on the other hand, the structure and properties of the real number system.

(From the course home page, my translation)

Although it may not be deliberate, the professor puts the development of the symbols and formalism competency before the mathematical thinking competency and the reasoning competency. When it comes to specific goals, the professor turns away from the competency description and focuses on the mathematical content instead.

## 5.2 Orally formulated learning goals

The professor was interviewed twice during the pilot study. He expressed his views about how he wanted to teach the course. This also included his beliefs about mathematical learning and his expectations for student participation and preparation for class. Before the main study began the professor was interviewed again, and he confirmed that he still had the same goals for his teaching. Furthermore, although the professor might have been influenced by participating in the pilot study, it is nevertheless reasonable to assume that his beliefs about learning did not change much. That is why in the following, I base the characterisation of the professor’s attitude on interviews from the pilot study.

### 5.2.1 Professor’s views on learning

In the pilot study interviews, the professor expressed the view that lectures should be a place where students *talk* about mathematics. At home they *read* mathematics and in the solving sessions and through the hand-in assignments they practice *writing* mathematics, and the lectures should then be a place where they *talk* about mathematics. According to the professor, this trisection (reading, writing and talking) is a rewarding way to approach the learning of mathematics.

Talking about mathematics can cover many different aspects. Often when students speak during lectures, it is to pose clarifying questions to the textbook or to the professor’s presentation, or to answer specific questions from the professor. The professor in this study expects more from the students than questions concerning difficulties with reading and understanding the textbook, or clarifying questions related to his review of the textbook content. The students should take part in dialogues about the kinds of questions one could pose to the subject matter, the range and limitations of the introduced concepts and questions related to the way the concepts are introduced. In the following, I call this kind of

discussion a meta-mathematical discussion (not to be confused with discussions concerning logic or other areas where the term meta-mathematical is used). The professor does not describe in more detail the nature of these discussions, and when and how they should be initiated, apart from explaining that the learning environment created should resemble his memories about the kind of discussions he had in his old study group at university.

The professor is not familiar with the terminology of mathematics education research, but if I relate his description of meta-mathematical discussions to the discussion in mathematics education research about mathematical understanding (see section 2.1.2), it is clear that he wants to develop the students' conceptual knowledge (Hiebert's definition) and relational understanding (Skemp's definition) through his teaching practice.

Besides engaging the students in meta-mathematical discussions, the professor also wants to show them that mathematics is not a bunch of facts that they have to memorise:

My goals concerning presenting proofs ... is that I feel that I owe it to the students once in their lifetime, or twice, to get the experience that it is not something they need to memorise, ... mathematics is actually something that one can think through almost from scratch. (Professor, pilot study)

The students should experience that the mathematical presentation is 'relative' – to a certain degree. That definitions and theorems can be stated in different ways, and that the presentation in the textbook is not the only way to present the subject matter. The professor believes that presenting different versions of definitions and theorems is a way to set the teaching apart from the textbook and present the subject matter from other angles, which is meant to enhance student understanding.

### 5.2.2 Professor's views on preparation

When preparing for class, the professor decides which definitions, theorems and proofs he definitely wants to present, but he prepares so that he is able to review all the proofs in a particular chapter. His aim is for student questions to influence what they talk about in class. In other words, the professor partly wants to base the lectures on those aspects of the theory that the students find difficult to understand while preparing for class. The reason for using 'partly' is that if the professor finds a certain result important, he would talk about it even if the students might not indicate that they experienced difficulties understanding it.

Although he is able to describe his expectations about student behaviour in class and their role in the teaching practice, he has a somewhat unarticulated idea about his expectations for student preparation and how the students could prepare in order to meet his expectations of having discussion-based lectures:

I don't know if I had hoped for any particular study behaviour, but I proba-



bly would have expected some kind of behaviour in which the student tried to read the textbook at home and had gone over the tasks and tried to solve some of them. (Professor, pilot study)

In his view, “*to read the textbook*” implies browsing through the results and noticing which parts are immediately understandable and which parts are troublesome. The students are not expected to spend hours trying to grasp one specific argument in a proof, as this, according to the professor, is a senseless way to spend time. In the second interview in the pilot study the professor implicitly describes his expectations for student preparation. He expects

... people [students] to have ... read it [the text] to a degree that one could begin to pose questions and be exposed to the concepts a bit. How far do they reach to find out what this concept means. Partly, what is the range, the limitations, what does it do, why don't we do it like this, wouldn't it be more logical or what are the pitfalls? (Professor, pilot study)

This description of the professor's expectations is at odds with a description of ‘the ideal situation’ made earlier in the interview:

You could say that the ideal would be to have a textbook where you could say that the student sits at home, reads it and understands it and then you could talk about it in class (he laughs). And it's difficult to find a textbook that makes it possible for students to do that ... [if the students had] read and understood the proofs, then there was no reason for me to review them one more time in class. (Professor, pilot study)

At this point in the interview I did not ask for an elaboration of his definition of understanding. I was sure that ‘understanding the text/proofs’ in this situation meant that the students understood the argumentation behind each deductive step in the proofs. With the danger of over-interpreting the professor's answers, what the professor is saying is that ‘talking about the text’ or having a meta-mathematical discussion about it is not a means to develop understanding. In this case, the professor expresses the view that ‘understanding it’ is not related to issues such as range and limitations of concepts, relationships between concepts, differences between definitions and theorems and so on.

This interpretation is further substantiated by his comments “... *then there was no reason for me to review them one more time in class*”. His review of a proof is meant to provide understanding, and since his proof reviews are very focused on the details (this claim will be supported in section 5.6), this is presumably what he means by understanding in this particular excerpt.

The different comments about his expectations of student preparation are contradictory. On the one hand, he does not want the students to mull over a minor detail in a proof, but on the other hand he seems to believe that understanding the proofs in detail is a prerequisite for having meta-mathematical discussions.

### 5.2.3 Professor's view on understanding

As just discussed, the meaning of ‘understanding the text’ could be that the argumentation behind each step in the proofs in the text has been validated. Later in the interview he describes another aspect of understanding:

...I believe, that is, my belief about how the mind works is that if you have several approaches to the same thing, then you remember it better, you understand it better. In one way or another you can say that understanding, what does it actually mean? Because much of it is just a question of ‘getting used to something’. (Professor, pilot study)

His view on understanding as a result of the process of “*getting used to something*” is reflected in other parts of the interview, for instance, when he talks about how students can benefit from seeing a proof that is actually too difficult for them to comprehend:

...maybe some of the proofs are too difficult, and even though they are reviewed this is not enough to actually understand them. But the fact that you have been engaged in the proof of a theorem is also sometimes the reason that you can remember the theorem afterwards. And maybe you don't immediately remember it but somewhere it has been stored in your memory that this theorem exists and then one day when you have to use it, then you have access to it much faster than if you had to come up with the theorem yourself and then had to go out and check if it has already been established. (Professor, pilot study)

The students should be able to remember theorems, and the act of reviewing proofs is a means to strengthen that memory. To “*store information in the memory*” requires that the information is placed in a cognitive network. According to Selden & Selden [1995], this can be achieved by strengthening the statement image of the theorem, and proof validation can be part of that process (see page 36).

The interviews show that the professor justifies some of his teaching activities with reference to goals which are not directly related to learning the particular mathematical content at issue (and not at all related to acquiring the mathematical competencies mentioned). For instance, in the last quote he explains that the purpose of presenting proofs that are too difficult for the students is that it might make them remember the theorems in the future. Even if seeing an incomprehensible proof might make the students more capable of remembering the result, it is not a goal in itself to be able to remember theorems if the theorems have not been understood, and it is doubtful that seeing an incomprehensible proof enhances the student understanding of the particular theorem. It thus seems that this activity does not follow directly from the written goals of the course.

After having looked at the professor's views on learning, understanding and preparation, it is now time to see how the actual teaching unfolded.

### 5.3 Elements of the teaching practice

What elements are essential in the characterisation of a teaching practice? In the pilot study an observation template was developed such that it contained elements describing the actual teaching practice, and this observation template was used to structure the observations in the main study and to provide an analysis of the teaching practice.

In order to present the analysis of the teaching practice, I have adopted a representation tool developed by Schoenfeld [1985] in connection with the analysis of solution processes. The core of the tool is to select relevant categories or elements and depict the evolution over time of the solution process, or in this case the teaching practice, according to these elements. Based on the observation template, I have chosen the following elements:

- **Extra-instructional activities, mathematical agenda** The professor's description of the lesson plan [Schoenfeld, 2000], information about homework assignments, preparation and so on. This group of activities mainly involves only the professor.
- **Concepts and mathematical structure** Comments, dialogues, and discussions concerning concepts and mathematical structure. Issues in the teaching practice which directly focus on conceptual knowledge and understanding. These activities can involve both the professor and the students, and can be initiated by both the professor or the students.
- **Motivating/illustrative examples** Examples or solutions to tasks provided by the professor where the aims are to explain the motivation behind or illustrate introduced concepts.
- **Repetition of results** Situations where the professor repeats previously introduced results (definitions and theorems).
- **Formal definitions and theorems** Situations where the professor presents and explains definitions and theorems.
- **Conventions** Comments from the professor concerning mathematical conventions, for instance that  $\delta$  and  $\epsilon$  indicate small margins in the domain and codomain, respectively.
- **Proof outline/proof validation** Periods where the professor outlines, reviews or validates proofs.
- **Solution strategies (including examples)** The professor's presentation of examples where focus is on solution strategies, or comments from the professor about solution strategies.
- **Task solving activities** Activities where the students try to solve textbook tasks or tasks provided by the professor, or activities where task solutions are reviewed by students or the professor.
- **Anecdotes/detours** Incidents where the professor takes a detour (from the textbook content), for instance of a historical or applicational nature.

In some of the ten teaching elements, both the professor and the students can initiate the activity, and participate actively in it. To be able to show when the professor and/or the students are involved in an activity, I introduce four different types of student/professor activity:

- **No student intervention**
- **Professor-initiated student activity**
- **Student-initiated professor and student activity**
- **Student-directed task solving activity**

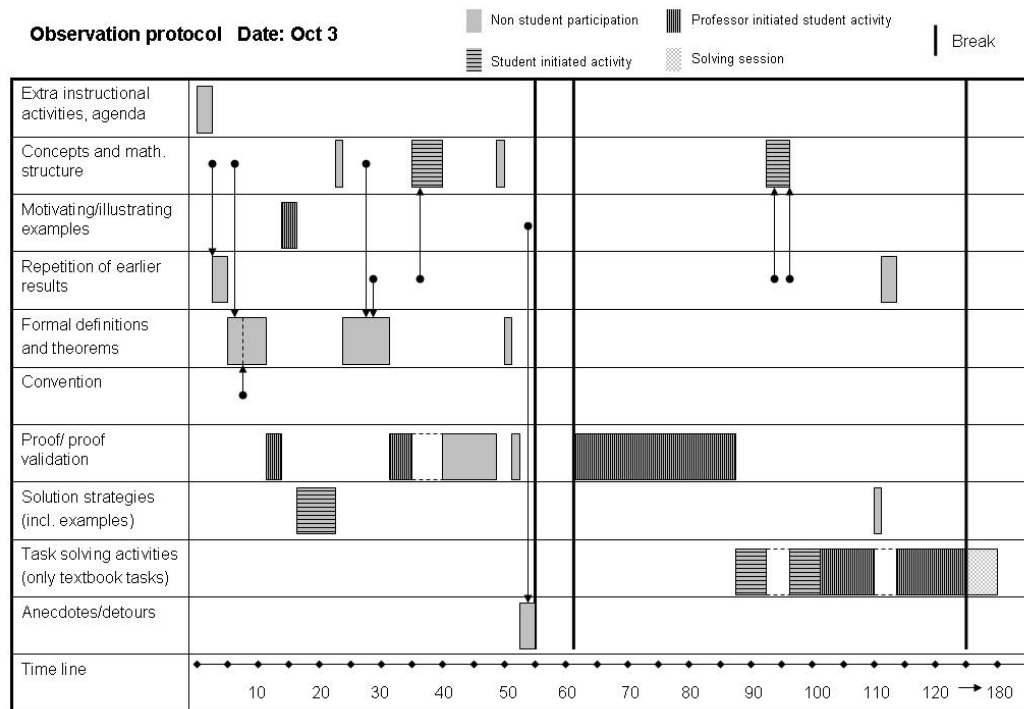
The first type of activity includes situations in which the professor talks in class without student intervention. The professor does not involve the students, for instance through questions, and the students do not interrupt to pose questions or offer comments. The second type describes situations in which the professor attempts to involve the students by, for instance posing questions or initiating student-student discussion. When the students pose questions and influence what the professor is talking about, the situations belong to the third type of activity. The fourth type only contains those parts of the lessons referred to as solving sessions, where the students choose which tasks to work on and the professor helps the people who ask for assistance.

### 5.3.1 Analysis of a lesson

Figure 5.1 shows the analysis of a lesson. The illustration may seem confusing at first, but I will go through the different periods and in that way explain the schedule. The lesson is divided in a lecture part (0-125 min) and a solving session part (125-180 min). The mathematical content of the lecture is limits of one-variable real functions.

The professor begins by giving some practical information about the hand-in assignments, and the home page, and continues to describe his agenda for the lesson (0-2.5 min). He starts by relating limits of sequences to limits of functions (the arrow), and then recapitulates the definition of a convergent sequence (2.5-5 min). Then he moves on to the formal definition of a limit of a function, compares it to the definition of limits of sequences (the arrow from ‘Concepts and mathematical structure’) and presents the first result related to limits of functions (5-11.25 min). During the presentation of the definition, a student asks a question about the notation (the arrow and dotted line). The professor tries to get the students involved in justifying the result he has presented, but ends up making a quick sketch of the proof himself (11.25-13.75 min).

Continuing with some illustrative examples, which are tasks from the textbook, the professor manages to involve the students in the solution (13.75-16.25 min). During this episode, a student ‘demands’ that the professor writes down the solution in a stringent way, which is why the episode changes and is classified under solution strategies with the pattern indicating student initiated activity (16.25-22.5 min).



**Figure 5.1** A time-line representation of the lesson. The lesson is composed of a lecture part (0-125 min.) and a solving session part (125-180 min.). Each period in the lesson has been characterised according to the ten elements, and the four types of student/professor activity. If the duration of an identified period is shorter than 1.25 minutes the period is symbolised by an arrow. Isolated questions from the professor or the students are depicted by full or dotted vertical lines, respectively.

After the examples, the professor returns to the relation between limits of sequences and limits of functions, and problematizes the transferability of the acquired knowledge regarding sequences (22.5-23.75 min). He presents a theorem that links the two domains, the sequential characterisation of limits (theorem 3.6), and explains that this can be used to translate what is known about sequences to also apply to functions (the arrow from ‘Concept and mathematical structure’) (23.75-31.25 min). This period also contains the repetition of an earlier result concerning sequences (the arrow). He intends to use the ‘translator theorem’ to prove an equivalent theorem for functions.

A period of proof validation follows (31.25-48.75 min), where a student’s question initiates an exchange about whether the image of a sequence is also a sequence (35-40 min). During this exchange the definition of a sequence is recapitulated (the arrow). After the intermezzo, the proof validation is continued, but without any student activity.

After small periods of talk concerning the relation between results (48.75-50

min), presentation of a new result (50-51.25 min), and a sketch of the associated proof (51.25-52.5 min), the professor ends the first part of the lecture with a historical anecdote (52.25-55 min) including an illustrative example (the arrow).

After the break, in the second part of the lecture, the professor goes through the proof of the sequential characterisation of limits (61.25-87.5 min). The professor initiates a dialogue, and the students attempt to participate (a closer analysis based on the notions of structure, components and details of this particular proof review will be provided later in section 5.9). In the beginning of the lesson, the professor had announced that after having talked about the chapter, they should discuss questions from the students regarding Cauchy sequences and the notions of limits supremum and infimum (topics from the previous chapter). A question from a student regarding one of the hand-in assignments, focuses the discussion around the solution to that particular task (87.5-101.25 min), with conceptual clarification and recollection (the two arrows) of sequences and subsequences. The professor asks if there are any questions concerning Cauchy sequences and limits supremum and infimum, and since the students remain silent, the professor asks the students to look at a particular task for 5 minutes and then they will go through it together (101.25-110 min). The presentation of the solution is run by the professor (110-125 min) and includes both attention to solving strategies (110-111.25 min) and recapitulation of limits infimum (111.25-113.75 min) – each time without student intervention.

After the presentation of the task solution the lecture part of the lesson is over, and the rest of the lesson (125-180 min) is devoted to task solving. The students choose if they want to work alone or in groups trying to solve textbook tasks and they are also free to choose what tasks they want to work on. The professor is available to answer questions from individual students or groups. His answers are never shared in public. Section 5.8 presents a more detailed analysis of the student-professor interventions taking place in this particular activity.

### 5.3.2 Comments concerning the analysis

Some comments about the teaching elements are needed. First of all, the order of the teaching elements is of course not unique. Another order might have been chosen, which would result in a different graphical image of the lessons. I have chosen this order, because I view it as ‘natural’. From my experience and observations of various mathematics courses during the PhD project, it seemed likely that a professor would start a lesson by giving some practical information, and presenting the agenda for the lesson. Then he might move on to talk about the concepts in the particular chapter, and might provide some examples illustrating the concepts. Before proceeding to formal definitions he might recall some previously introduced concepts. After stating definitions and maybe explaining convention-oriented issues relating to definitions, the professor would move on to theorems, and provide associated proofs. Then periods of task solving might come in, where the professor could focus on how to solve tasks using the presented

definitions and theorems, and also go through solutions to specific tasks from the textbook. The last teaching element is by its very nature ‘a detour’, and thus I found it most natural to place it last.

Reality is not always so regular as the above description and the proposed ten teaching elements suggest. There are situations where it can be difficult to determine whether the professor just explains a particular definition or he actually addresses issues that could be categorised as relating to the mathematical structure, for instance touching on relations between concepts. The following excerpt corresponds to the time span (3.75-7.5 min), where the professor repeats the definition of a convergent sequence and presents the definition of the limit of a function:

**Professor:** Now that you're experts in limits of sequences it can't come as a surprise how you should define limits of functions. Or can it? If we have a sequence tending to some element  $a$  then we usually say that we have to be able to – given some epsilon, an epsilon-window around  $a$  – find a capital  $N$ , such that all the subsequent elements of the sequence lie within this window. When we talk about functions, we have some  $x_0$  here, here we have a number line, here we have an  $a$ . I don't say anything about the function being defined in  $x_0$  or not. Then you could ask if the limit of  $f(x)$  could be the same (meaning: defined in the same way)?

We have a window, an epsilon, plus epsilon, minus epsilon around  $a$ , then we can, if we just get sufficiently close to  $x_0$  secure that all the function values lie within this window. And to lie close to  $x_0$ , we can characterise that by an inequality. So that must mean that we have a function  $f$  on some interval, not inclusive the point  $x_0$  – it's not such that  $f$  can't be defined in  $x_0$ , I just don't care if it's defined or not – to  $\mathbb{R}$ . Then  $f(x)$  approaches  $a$  when  $x$  approaches  $x_0$  (he writes:  $f(x) \rightarrow a$  with  $x \rightarrow x_0$  under the arrow).

So you could say the limit for  $x$  approaching  $x_0$  for  $f(x)$  equals  $a$  (he writes:  $\lim_{x \rightarrow x_0} f(x) = a$ ), if and only if for every epsilon larger than zero there exists, yes, now it's not the  $n$ 's, now the  $x$ 's have to lie close to  $x_0$ , and lie close that could for instance be described by distances, there exists a delta larger than zero, such that for all  $x$  in this interval for which 0 is smaller than the distance from  $x$  to  $x_0$ , smaller than delta, then  $f(x)$  minus  $a$ , absolute value, has to be smaller than epsilon (he writes:  $\forall x \in I : 0 < |x - x_0| < \delta : |f(x) - a| < \epsilon$ ). That looks like the same, right? Instead of pushing forward some capital  $N$ , we push the values close to  $x_0$ , no what should we say – the variables close to  $x_0$  and then it has to push the function values close to  $a$ .

In the excerpt, the professor states the formal definition in lines 19-26. What comes before is a repetition of the definition of the limit of a sequence and an explanation of the definition of the limit of a function accompanied by an illustration. During and after the formal definition, the professor makes comments about the connection between the definition of a convergent sequence and the limit of a function (“yes, now it's not the  $n$ 's”, line 21 and “instead of pushing forward

*some capital N*”, line 27). These two comments are illustrated with one arrow in the time-line representation of the lesson. The first part of the excerpt could be categorised as ‘Concept and mathematical structure’, but since the professor does not go beyond a repetition and an explanation of the two definitions, I have categorised the excerpt as ‘Repetition of results’ first (lines 1-10) and hereafter as ‘Formal definitions and theorems’ (lines 11-29).

Another point that can cause difficulties in the categorisation is to identify when solution strategies are being addressed. The following excerpt covers the time span 13.75-17.5 min (motivating example and part of the following period of solving strategies).

- Professor:** Eh, let’s try some examples . . . try to look at b in task 1. 1b. (The professor writes down the task on the blackboard:  $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = 2$ ). Why? Why is it, if you look at the fraction, try to look.  $x^2$  minus 1, that tends to zero, right, when  $x$  tends to 1. And  $x$  minus 1, that tends to zero. It is a fraction of the type, zero divided by zero, that is not possible, is it? Benny, what do you say, can we say something about the limit after all?
- Benny:** We can begin by rewriting it.
- Professor:** Okay.
- Benny:** So it says  $x$  minus 1 multiplied by  $x$  plus 1 in the numerator. And then divide the denominator out.
- Professor:** (The professor writes it down) We can simply reduce here. Then it just says  $x$  plus 1. If we have two equal functions, away from 1, namely this  $x$  plus 1 and this fraction. The function  $x$  plus 1, it’s not so difficult to figure out where this tends when  $x$  approaches 1, of course it approaches 2, but according to remark 3.4 this function also tends (points at the original fraction) to 2. . . Do you remember some lessons ago..(Brian interrupts)
- Brian:** Have we then proved it by using definition 3.1?
- Professor:** Almost. (writes down) Given epsilon . . . larger than zero, choose a delta larger than zero, such that for all, eh,  $x$ , zero smaller than the distance from  $x$  to 1 smaller than delta,  $x$  minus 1, absolute value, has to be smaller than epsilon. We can do that, that is the definition that  $x$  approaches 1, right? Let’s look at, let’s take a delta, we take the same delta. (Writes down) If zero is smaller than  $x$  minus 1 smaller than delta, then  $x$  plus 1 minus 2, absolute value, then the distance between these two, that is the difference between  $x$  and this amounts to 1 which is smaller than epsilon (he writes:  $|x+1-2| = |x-1| < \epsilon$ ). So I have reduced the problem to definition 3.1 here. I have just applied, 3.1 assures me that given an epsilon I can choose a delta so that this up here is fulfilled and that is what I use here.

When Brian asks if they have used the definition to prove the result (line 19), the professor shifts from using the task as an illustration of the theorem to focus on applying the definition of the limit of a function. It can be argued that the first part of the excerpt also concerns solving strategies, namely to apply remark 3.4. In my view, the focus in the first part is not on generally applicable solving



strategies as such but instead on obtaining a result of the particular task under consideration, whereas the professor in the second part of the excerpt focuses on how to use the definition to obtain the result.

Repetition of results only includes those situations where the professor (or a student) mentions a previous result and repeats or explains it in some detail. If the professor says, for instance, “*here we have used definition 3.1*”, without making any recollection of what definition 3.1 says, it has not been classified as a repetition of results.

### 5.3.3 Results of the analysis

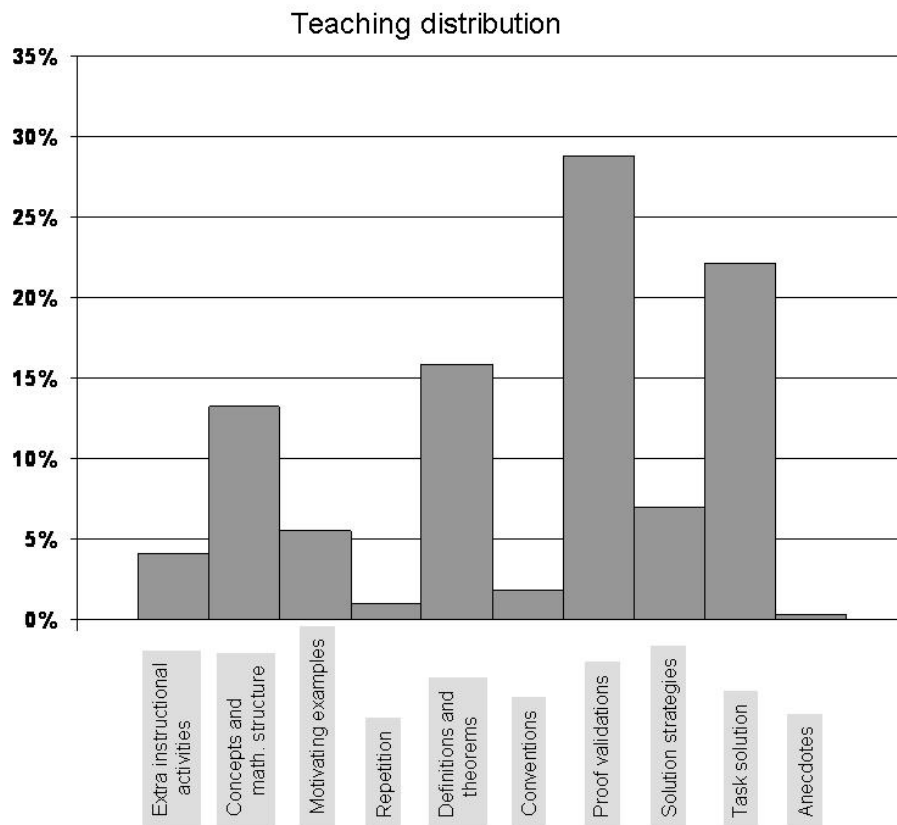
Time-line representations of twenty-five lessons are listed in appendix B.1. When the representations are compared, it becomes clear that the lessons are very different. The lessons do not follow strictly the stipulated process of a prototypical lesson, described in the previous section, which provided the rationale behind the order of the ten teaching elements. Instead, there are lots of smaller periods and the professor or the students initiate breaks so that the teaching jumps more or less randomly between the elements.

In the following I will only address the lecture part of the lessons, while this is the part that the characterisation concerns.

One thing the lectures have in common is that the periods of time devoted to talking about or addressing issues concerning concepts and mathematical structure are rather short and often appear as intermezzos in periods of either presentation of definitions or proof validation periods, and often when the professor is erasing the blackboard. There are of course exceptions, for instance the lesson on the 6th of October, where twenty minutes were spent talking about how the limits of functions can be defined using the topological notion of a neighbourhood. Since topology is not part of the course syllabus (the professor had mentioned the concept in a previous lecture), the students were confused about what the professor was talking about and the purpose of having two definitions of limits of functions. The professor’s goal was clearly to introduce the notion of a neighbourhood to the students in the hope that having seen it before would make it easier for them to learn it properly in a later course.

How could classroom discussions concerning concepts and/or mathematical structure otherwise be initiated? I would like to give an example from a different teaching practice such that the reader will get a more clear idea about the content of this teaching element. Another professor told me how he initiated a classroom discussion about the range of the definition of continuity. Since this particular professor had experienced that many students confuse epsilon and delta in the definition of continuity, he discussed with the students what kind of functions fulfilled a different definition where the delta is provided and epsilon is to be chosen (the students had prepared at home to discuss this). This discussion aimed to make the students realise the importance of the details in the definition and the consequence of not getting them right, aimed to provide them with

an experience of what is entailed by making new definitions and examining the consequences of them. These are issues that I would characterise as belonging to ‘concepts and mathematical structure’.



**Figure 5.2** The table shows the average distribution of the ten main elements in the lecture part of a lesson. Solving strategies occupy almost 7 percent of the collected lecture time for the twenty-five lessons, but this is mostly due to the lesson on December 15. If this lesson is removed from the distribution, solving strategies would only occupy 3 percent of the collected lecture time.

Figure 5.2 shows the average distribution of the ten different elements of the lecture part of a lesson. From this distribution other striking characteristics of the lectures are visible. Much time is devoted to proof validation, and also to presentation and discussion (where the whole class participate) about task solutions from the textbook. When looking at the time devoted to solution strategies, this portion is surprisingly small, just below 7%, and this result is mostly obtained by the lesson on the 15th of December, where the professor spent the entire lecture talking about and exemplifying five different proof strategies.

The lecture part of the teaching practice is thus primarily focused on defini-

tions, theorems, proofs and task solutions, while solution strategies come second. This might not be a deliberate choice of the professor, in fact a comment provided by the professor before the presentation of a proof, showed that he considers proof validation as an activity which provides the students with solution strategies.

The analysis shows that the professor seldom repeats earlier definitions or results. He mentions earlier introduced definitions during introductions of new concepts and during proof reviews, but often without recapitulating the essentials. When periods of repetition do occur, they are often short and initiated by the students.

This way of analysing the teaching practice focuses on the composition of the teaching practice, i.e. on the elements therein. The next section concerns the norms that are established in the classroom, both the norms that the professor explicitly tries to establish and also those that seem to be established unintentionally.

## 5.4 Establishment of social norms

The following characterisation is grounded in the framework of Cobb and co-workers about the establishment of social and sociomathematical norms in the classroom (described on page 41).

I have detected two different groups of social norms in the classroom observed in the main study. One concerns the norms of participation or norms of discourse and the other concerns norms of preparation. I have defined them as social norms, but they are not completely unrelated to the fact that they concern a mathematics classroom.

### 5.4.1 Norms of participation

Already in the professor's presentation of the course in the first lesson, he gives the impression that the progress of the course will be influenced by the students' ability to understand the content. He does not have a completely finished and unchangeable course plan. Instead he will wait and see how the students cope with the content acquisition. This implies that the students are expected to reveal if they have trouble understanding something or if the professor progresses too fast or too slowly.

Classroom observations support this. The professor often tries to involve the students by asking questions, sometimes directed to the whole group of students and sometimes directly to an individual student. In the following excerpt, the professor asks the class if they are able to translate a string of mathematical symbols (including quantifiers):

- 2      **Professor:** Should we write down in words what it says (points at the formal definition:  $\forall \epsilon > 0, \exists N \in \mathbb{N} : n > N \Rightarrow |x_n - a| < \epsilon$ ) or are you gradually able to read a sentence like this? (the students are busy

- 4 writing down). . . . Who can read this? (Two students raise their hands)  
 Does this mean that the rest of you don't know how to translate this  
 6 if you met it in the book?
- Carrie:** Almost.
- 8 **Professor:** Almost.
- Carrie:** It is almost there.
- 10 **Professor:** It's almost there (other students agree). From this you can, you  
 can read it like 'for all' or 'for every' epsilon larger than zero and  
 12 either 'there exists' or 'there is', a capital  $N$  in the natural numbers  
 and then I will read this as 'such that' for every  $n$  larger than capital  
 14  $N$  then the distance, the absolute value of the difference between  $x_n$   
 and  $a$ , that is the distance from  $x_n$  to  $a$ , is smaller than epsilon.
- 16 **Iris:** That full stop?
- Professor:** Is there a full stop? There shouldn't be (erases something after  
 18 epsilon)
- Bob:** No, the one in the middle.
- 20 **Professor:** That is a colon (makes the colon more clear).

Since the majority of the students do not yet feel familiar with the translation of the mathematical formalism, the professor willingly explains it. The students are also expected to follow and participate in the professor's review of the textbook content, and to listen to and comment on other students' questions, ideas and justification attempts. The norm that the students are expected to say if they do not understand what the professor is saying and the norm that the students are expected to participate actively in the professor's review of the textbook content are both exemplified in the following excerpt from the beginning of a lesson concerning continuity a month after the course started:

- Professor:** Can anyone say in a few words what the difference is between  
 2 continuity and what we have learned about limits? Or what continuity  
 has to do with limits? (He turns his back to the students and continues  
 4 to erase the blackboard. A student raises his hand)
- Professor:** Can you try? What is your name?
- 6 **Tyson:** Tyson.
- Professor:** Tyson.
- 8 **Tyson:** It has to tend to the same value if you approach it from the right or  
 the left. If it tends to the same value if you approach it from the right  
 10 or the left then it's continuous.
- Professor:** What do you say to that interpretation? . . . What is the difference  
 12 then from limits? That is also something about approaching the  
 same value from right and left.
- 14 **Tyson:** Can't it have, it can have a limit value at the point without being  
 continuous.
- 16 **Professor:** It can, yes, yes.
- Student:** It can also be a final point.
- 18 **Professor:** It can also be a final point, yes. But it can also, it can have  
 a limit without the functions being continuous without being defined.  
 20 So what is new about continuity. Marie?
- Marie:** That the function is defined in the point  $x$  and the value of the  
 22 function is equal to the limit value.

24       **Professor:** And the function value is equal to the limit value. So we not  
                   only have a limit value, it's also the value of the function in that point.  
 26       Continuity is where you link the two things. (the professor continues  
                   to erase the blackboard)

From the excerpt, it is clear that the two norms have already been established in the course at this point. Some students find these norms supporting of their learning, and that the professor is creating an informal and constructive learning environment:

I think it's good that he asks, so people ask, like about everything, right? It's all contributing to putting things into perspective. So I wouldn't be without that. And this is one of the things that are nice, that there is so much dialogue, more talk. Even if you are wrong, right? Like last time I said something completely wrong, but I didn't feel, like sometimes, I just feel that it's nice that there is room to say this is how I see it and that you can say it without thinking: No, this I don't dare to say. I think that is nice. So I think that the atmosphere where he asks and we ask back instead of him just going on, but he includes us, that is very good. It gives an informal..I think that's the right way to do it, for my part at least. (Dennis, student)

Some students find such a teaching style intimidating and disturbing and would prefer that a professor just talked without involving students:

I don't like teachers who pose questions directly to the students, 'would you please answer this', questions like that. I don't like that at all. That stresses. It has to be his job [to present the subject matter], and then if you feel like it you can intervene. (Matthew, student)

The norms of participation established in the classroom are related to the belief that mathematics is learned through discussion where arguments are challenged, accepted or refuted. Other subject matters, for instance medicine, have not the same belief about learning attached to them, so the same norms of participation would probably not be established to the same extent in those classrooms.

#### 5.4.2 Norms of preparation

The homework for each session was available on the course home page and would typically consist of a chapter in the textbook of about 7 pages and about 9 textbook tasks (the assigned homework is listed in appendix A). In the quote on page 150, the professor describes that the 'ideal' would be that the students were able to understand the text at home and then the lectures could be used to discuss the content. This 'ideal' is shared by most of the students. One student expresses it as: "*This was my perception of university, then you had read it and then you would have understood it and then you could take it from there when you joined the class*" (Carl, student). But there is, as always, a difference between ideal and reality. In the classroom a study habit/preparation norm different from

the ideal preparation norm was established during the course, starting with the professor's introduction at the first lesson:

- Professor:** I use to say something about how you should prepare for class.  
 2        You read the text before class and you try to solve the assigned tasks.  
           You will probably discover that it isn't all the tasks that you will be  
 4        able to solve but then it's a good thing that we can meet in class and  
           discuss them. I will from time to time present proofs but it's the idea  
 6        that you should learn to read and understand proofs yourselves. I will  
           probably have to help you a lot in the beginning.

This introduction, in my opinion, does not indicate that the professor expects the students to have fully comprehended each step in the proofs – it is an aim that they will be able to fulfil at the end of the course, but not a prerequisite for participation in the lectures. During the lectures the professor sometimes gives recommendations for preparation that contradict his 'ideal' situation. In the following excerpt from the fourth lesson, the professor is presenting an example from the textbook, and asks the students for the choice of a capital  $N$  in an epsilon- $N$  proof:

- Professor:** So the question is, how do we find such a capital  $N$ ? (he erases  
 2        the blackboard) Now, this example is in the textbook, so it might be  
           an advantage to have read it at home, then you would know what to  
 4        do ... I'm not going to say what you should do, but what you could  
           do.

Although the professor might be a bit ironic here, he is not sending a clear message that the students are expected to struggle to understand the text before going to class. A couple of weeks later, the professor sends a similar signal regarding the expectations to the students' preparation:

- Professor:** Instead of going into the proofs, I would rather spend some time  
 2        talking about what the theorems say and then leave the proofs up to  
           you. And let's hope that the next time we meet you will say 'can't we  
 4        spend some time on this proof?'

The professor indicates that he does not expect the students to have tried particularly hard to understand the proofs in the chapter before he explains them. Instead he indicates that the students are supposed to read the chapter *after* his presentation and not before. These small comments contradict the 'ideal' that the students and the professor shared when both groups were interviewed. So the social norm about preparation as the professor, more or less intentionally, tried to establish in the classroom was that it would be 'a good idea' to acquaint oneself with the chapter *before* a lesson, but close-reading was expected to take place *after* the lesson. The professor thus sends the message that it is not necessary to have prepared thoroughly before a lecture in order to participate in it, but that the students of course are expected to study at home.

The results from the preparation habit study, which will be presented in the next section, do not clearly indicate whether this norm of the professor was in fact established in the class.

### 5.4.3 Students' preparation habits

The presentation of the results from the study of the students' preparation habits is organised in two parts, reading and task solving, where task solving both concerns the time spent and number of tasks solved. The preparation log is listed in appendix H.

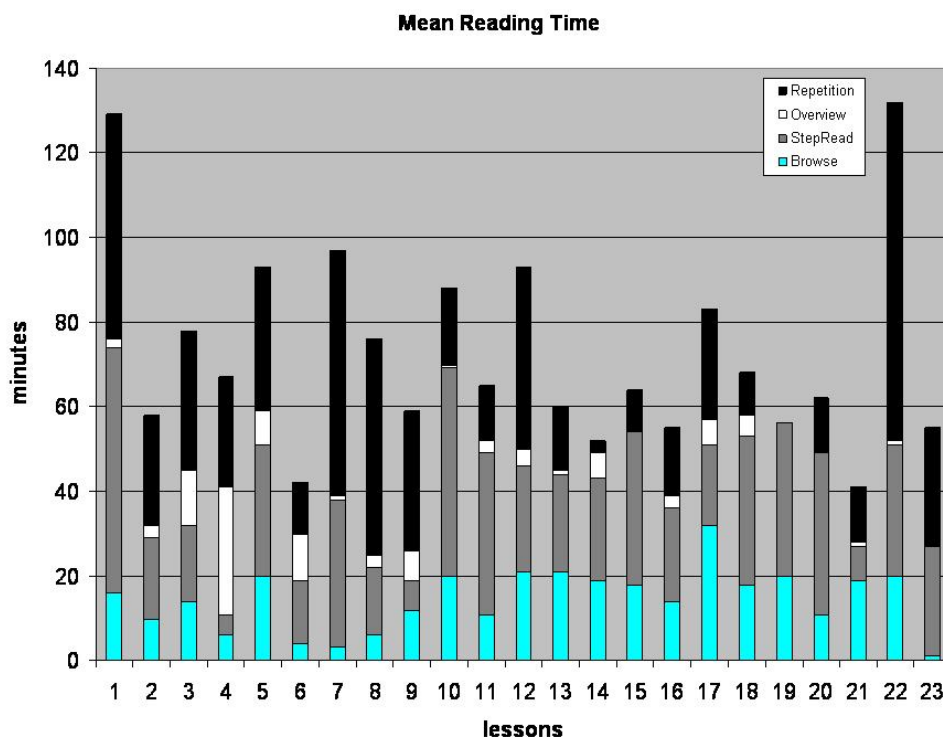
#### 5.4.3.1 Reading

The students were asked to distinguish between four different categories of reading. The first category encompasses students' browsing through the text without trying to understand each step in the proofs, and the second category covers the time they spend trying to understand each step in the proofs and in the examples. When the students try to figure out what are the important results and how the results are linked, or when they try to develop conceptual understanding, the time should be registered under the third category, while the fourth category considers the time they spend reading previous chapters or material from previous courses (for instance prior calculus courses).

If the ideal teaching practice that the professor talks about in the interviews, should stand a chance of being established, it would require that most students spend a considerable amount of time on the second and third category of reading. The preparation norm that the professor is tending to through his preparation guidelines suggests that the students spend a lot of time on the first and fourth category.

There seem to be two characteristic term intervals during the 23 lessons, see figure 5.3. The first period consists of the lesson 1 to 9 and the second is from lesson 10 to 23. There are two distinct outliers, lesson 1 and 23, caused by a single student each time (although not the same student in both cases), and thus not evidence of a general pattern that several students chose to spend more time reading for those two particular lessons.

In the first period, the students spend about half of their reading time revising previous chapters, whereas the time spend on revision is diminished in the second period and the time spend on browsing and comprehending each proof step is increased. In the first period, the students behave according to the professor's indications in class: they quickly browse through the text (average: 10 minutes), trying to create an overview (average: 9 minutes) and grasp the details in the proofs (average: 23 minutes). The rest of the time is devoted to reading previous chapters (average: 36 minutes). Then there is a shift to the second period where they now use 18 minutes to browse, 2 minutes to form an overview, 29 minutes to read proof details and 21 minutes to revise previous chapters.



**Figure 5.3** The table shows the average preparation time, across students, in minutes devoted to reading before each lesson. The four categories are *browsing through the text*, *trying to comprehend each step in the proofs*, *trying to get an overview of the text*, and *revision of previous chapters*. There are two distinct outliers at lesson 1 and 23.

The change in preparation habits may result from an increase in the level of difficulty of the textbook proofs. The change can also be caused by the fact that the professor gradually spend more time reviewing proofs in class (maybe because the proofs are getting more complicated), so the students experience that this is the most important aspect of the teaching and that they have to study the proofs more carefully in order to be able to understand what is going on in the lectures. When looking at the student distribution (not shown), the students are distributed around the average value in the first period, whereas they are distributed around two values in the second period. Here one group of students spend very little time on preparation (probably because these students had projects to complete), and the remaining students in return increase their preparation time. Since the professor sends the signal that it is possible to participate in the lectures without having prepared, it could be a contributing factor to the students' choice not to prepare when other deadlines start to pressure them.

On average, 73 minutes of the students' preparation time is devoted to read-



ing, and 47 of these minutes are spent on the chapter that the professor is going to review. Taking into consideration that the material is new to the students, and that a chapter contains three to four theorems and associated proofs, 47 minutes is not much. It is reasonable to assume that the students are not prepared in a way that enables them to conduct meta-mathematical discussions about concepts, definitions and results.

#### **5.4.3.2 Task solving**

The two different term intervals do not reflect in the amount of time the students spend on task solving, and in fact there is no significant change in the average amount of time the students spend on task solving during the course.

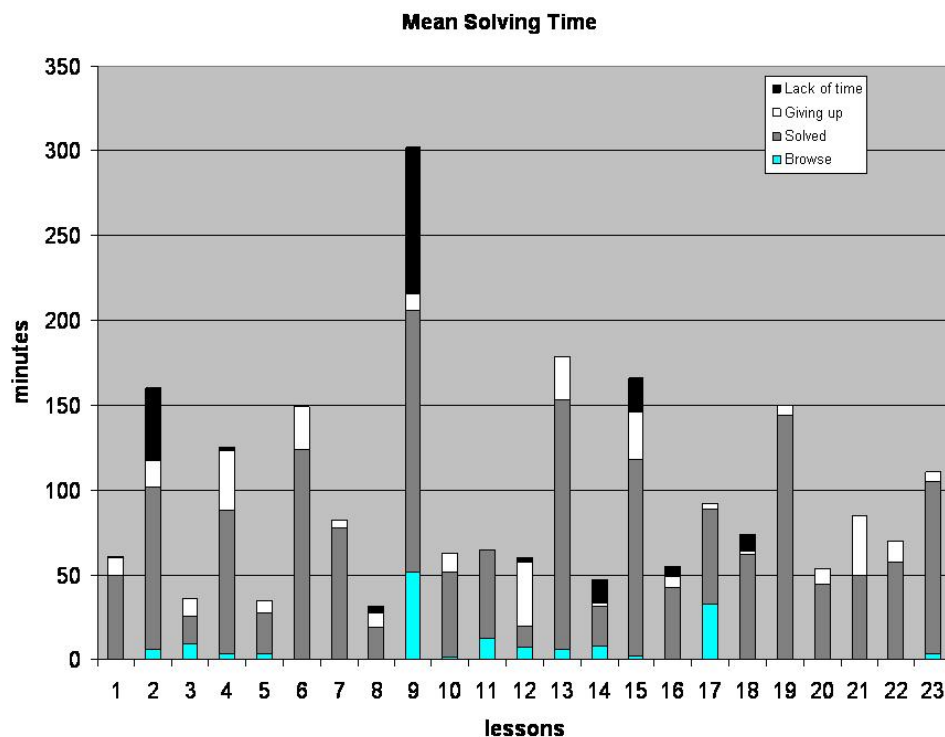
There are also four different categories regarding task solving. When the students look at the tasks in order to get a first impression of what is easy and what might cause troubles, the time spent and how many tasks they looked at should be registered in the first category. In the second category the students note the number of tasks they believe they have solved and how many minutes they spent trying to solve them. The third and fourth category consider the tasks that they gave up solving either because they could not figure out how to solve them or because they ran out of time. Furthermore, the students have to specify if they solved the task alone or in a group.

On average, they spent 52 minutes working alone on tasks where they reached a solution and 18 minutes in groups, and on average that amounted to 1.6 solved tasks and 0.4 solved task in groups. The results are depicted in figure 5.4 and figure 5.5. From this result it seems that each student on average only manages to solve two tasks from each chapter, but the preparation log does not include the number of tasks they work on in the solving sessions, where the professor is available to help them.

#### **5.4.3.3 Combining reading and task solving**

On average, the students spent almost three hours (171 minutes) preparing before each (three hour long) lesson, where 1hour 13 minutes were spent on reading and 1hour 38 minutes were spent on task solving which on average resulted in two solved tasks. It is of course natural and relevant to ask if the amount of preparation time and the number of solved tasks are sufficient, but it is not straight forward to judge this.

Students at this level and at this particular university are expected to take two courses and work on one project in each semester. The project work supposedly consumes half of their study time. So if they take two courses with 12 hours lesson time and 12 hours of preparation per week, this adds up to half of their study time in a week, thus amounting to a 48-hour study week. In Denmark, a workweek is 37 hours, but university students are usually told that they should expect a longer workweek since they have a long summer break. In that light, a 48-hour study-week seems fairly reasonable.

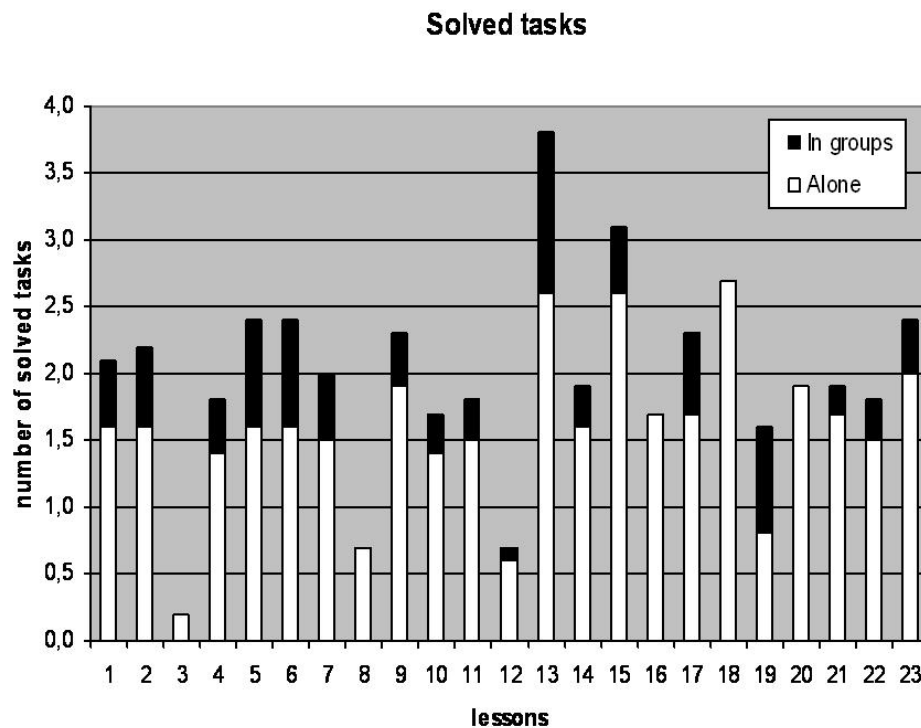


**Figure 5.4** The table shows the average preparation time per student in minutes devoted to task solving. The four categories are *browsing the assigned tasks*, *reaching a solution*, *giving up due to lack of solving abilities*, and *giving up due to lack of time*. The table does not separate between task solving alone or in groups. There is one distinct outlier at lesson 9 due to one student and not to a general pattern for that lesson.

There is another way to evaluate the time consumption. In Denmark, external lecturers are expected and get paid to spend 2.5 hour on preparation for every class hour [Finansministeriet, 2001]. And he or she already knows the subject matter. The students do not know the content so it would be reasonable to expect them to spend more time than the person teaching them. From this point of view the students' amount of preparation time appears severely short.

#### 5.4.3.4 Reliability of the preparation survey

In the student interviews, the students were asked about the preparation log that they had filled out during the course. The purpose was to evaluate the (methodological) reliability of the survey. Out of the ten students, only four answered that they had not experienced any difficulties filling out the preparation logs. So even though the students had participated in the design of the preparation log, many of them still experienced difficulties filling it out.



**Figure 5.5** The table shows the average number of solved tasks per student before each lesson. The table distinguishes between task solving done alone and in groups.

In the hope of ensuring high reliability, I had explained the different categories to the students, and handed out written explanation which they could look at whenever they needed to. In spite of these efforts, some students still found it difficult to remember and check the meaning of the different categories. A student expressed that

..I don't find it [the log] 100 percent obvious, but that's because I often don't get past the first box. The problem is that I haven't read this log carefully, so every time I am surprised by the content of the different boxes. ...for instance, the first here 'browsing tasks', there I'll always end up writing how long time I spend trying to solve them. (Carl, student)

Another student stated that *"I know that when you make a questionnaire like this it's just one suggestion for how things might be, so I don't take it so seriously if it fits into your categories or not. ...I approximate as well as I can ...I just fill it out and don't worry much about it"* (Carrie, student).

Some students studied the textbook in a way that combined several of the categories and they found it difficult to register the time spend on each category.

A student described it this way:

...I read, browse the text and there are some things that you immediately understand. Then there is something that you look at one more time – and now you’re not browsing. But it’s difficult to separate what it is that I really work through, because it happens concurrently. And also because I usually fill out the log in class. And especially with the tasks, because when I sit in a group, I at the same time try to solve some of the other tasks beside the one that the group tries to solve. (Benny, student)

From the students’ replies in the interviews it became clear that not all of them had understood what the third reading category actually implied. A student revealed that *“for me, to browse and to get an overview of what is major and minor is basically the same thing.”* (Aaron, student).

These difficulties have an influence on the accuracy of the results of the preparation survey, but the inaccuracy is mostly confined to separation of the categories and not to the total time spent. This means that the total reading time is more trustworthy than the time spend in each of the four categories.

## 5.5 Establishment of sociomathematical norms

Three sociomathematical norms seem to be in play in the classroom. The professor tries to establish the norm that the students should act as ‘mathematicians in spe’ (prospective mathematicians), but it is difficult for the students to do what it takes to take this norm on board. The second norm concerns what is mathematically accepted forms of argumentation, and the third norm concerns the use of tricks in proof validation and proof construction situations.

### 5.5.1 ‘Mathematicians in spe’

The professor attempts to engage the students in activities characteristic of mathematical enterprise (this was also established in the professor interviews), and this especially concerns engaging in mathematical discussions and argumentation processes, when proofs are being validated. In spite of his endeavours, the students do not manage to participate in a way that makes this norm an established norm in the classroom. The following excerpt shows what usually happens when the professor tries to engage the students in proof validation. The situation takes place at the end of the course, and concerns the validation of the proof of the statement that taking limits and integrating are interchangeable for a uniformly convergent sequence of functions (theorem 7.10):

- Professor:** First of all, when we talk about being Riemann integrable, what  
 2        does it take to be Riemann integrable? Aaron?
- Aaron:** You have to be bounded.
- Professor:** You have to be a bounded function. And? That was one of the  
 4        conditions. There is one more. Oscar?

- 6     **Oscar:** We look at a closed interval.  
       **Professor:** Yes. But what is the definition of being Riemann integrable  
 8       besides being bounded. There are two things. One is boundedness,  
       the other is?  
 10    **Bob:** Monotone?  
       **Professor:** You say monotone? I thought you said upper sum. What do you  
 12       say, Marie?  
       **Marie:** I don't say anything.  
 14    **Professor:** Ryan?  
       **Ryan:** The lower sum has to be equal to the upper sum.  
 16    **Professor:** Almost, right? Except epsilon. If we have an epsilon, then we  
       have to find a partition such that the difference between the upper sum  
 18       and lower sum for that partition is less than epsilon. So what we have  
       to do, what's it called, account for, is that  $f$  is bounded and we have  
 20       to point out a partition such that the difference between the lower sum  
       for  $f$  and the upper sum for  $f$  is smaller than epsilon. That is our task.  
 22       Is  $f$  bounded? Why? (10 sec. pause)  
       **Tyson:** Because it's uniformly convergent. (he is presumably taking about  
 24       the sequence)  
       **Professor:** Yes, and?  
 26    **Tyson:** And  $f_n$  is Riemann integrable and bounded, then  $f$  will be too.  
       **Professor:** Do you buy that? (10 sec. pause) Can you make it more explicit?

The professor is (only) trying to make the students repeat the definition of Riemann integrability. At last he receives an almost correct answer (line 15) but is forced to provide the formal definition himself. The episode takes 7 minutes. Since this step is only a minor step in the proof, it is not so surprising that he chooses to carry out the rest of the proof without involving the students. The rest of the proof takes 35 minutes.

### 5.5.2 Accepted forms of argumentation

The professor tries to establish the sociomathematical norm in the classroom that mathematically acceptable argumentation is founded on the notions, concepts, and results introduced in the course. It is not acceptable to base argumentation on prior experiences, hunches, intuition, and such like. In the following excerpt, the professor is trying to get the students to construct a proof using the definition of limits of functions. The excerpt begins where he states the theorem to be proven (remark 3.4 in appendix C.2):

- 2     **Professor:** .. if you have two equal functions except at  $x_0$ , and if one of  
       them has a limit then the other one has a limit too and it's the same  
       one. You also think that this is clear, Carl?  
 4     **Carl:** Yeah, yeah.  
       **Professor:** Does it also clearly follow from the definition over here?  
 6     **Carl:** Now I have been preparing for today, so ..  
       **Professor:** Oh, you have, ... Annie, is it clear?  
 8     **Annie:** Do I think it's clear?  
       **Professor:** Yes.

- 10     **Annie:** This is the time where I should say that it's totally clear, or?  
        **Professor:** That depends how you feel, right?  
 12     **Annie:** So, I am a bit hesitant. It all needs to seep through.  
        **Professor:** Carl, you have no doubts, why not?  
 14     **Carl:** I learnt it in high school.  
        **Professor:** You learnt it in high school? (Students and professor laugh) That  
 16         is not a good argument.

The professor indicates that to think that something “*is clear*” needs to be justified, and in this case a proper justification is based on the definition of a limit of a function (line 5). He shows the class (line 15) that it is not acceptable to use arguments such as “*I learnt it in high school*” (line 14). It is difficult for the professor to establish the norm in the classroom that argumentation has to be based on definitions or already proven results. The previous excerpt showed that still late in the course, the students were reluctant to use (or unable to remember) a definition to justify why a given object possesses a certain property.

### 5.5.3 The role of tricks in proof validation and proof construction

In the pilot study interviews, the professor expressed that learning and understanding mathematics also have to do with experience and “*getting used to*”. When the professor makes a certain deduction, introduces a particular entity, or uses a specific result, the students often perceive this as the use of tricks and good ideas. Especially, in epsilon-delta (or epsilon-N) proofs, some students express that for them the solutions are based on good ideas:

...to be able to find those deltas and epsilons [demands good ideas]. ... that is how I perceive it. The professor tries to break up [the proofs] to something more simple, right, but sometimes I just feel that I couldn't have done it myself. That is, to get this idea to put delta equal to something particular. This is how I felt most of the times. (George, student)

In the following excerpt, the professor is trying to help two students with the task<sup>2</sup>: If  $f : [0, \infty[ \rightarrow \mathbb{R}$  is continuous, find  $\frac{d}{dx} \int_1^{x^2} f(t) dt$ . They were stuck and had decided to ask the professor for help:

- 2     **Professor:** What if we try to introduce an auxiliary function as the integral  
        from 1 to  $y$ ?  
        **Annie:** We know that  $y$  is  $x^2$ .  
 4     **Professor:** Yes, but let's try to write down the integral from 1 to  $y$ , instead,  
        of  $f(t) dt$ . We could call it  $F$ , couldn't we? What if we take  $F(x^2)$ ,  
 6         what do we get then?  
        **Annie:** If we take what?

<sup>2</sup> The task (5.3.3a in [Wade, 2004, p. 134]) is associated with a chapter including the *Fundamental Theorem of Calculus* and the theorem of *Change of Variables*, theorem 5.28 and 5.34 in appendix C.2.

So instead of pointing the students in the direction of the Fundamental Theorem of Calculus and the theorem of *Change of Variables*, which presumably would not have seemed so mysterious to them, the professor uses what seems to be perceived as a trick by the students.

The norm that task solutions often are founded on tricks and good ideas was not only established during the solving sessions and during the episodes where solutions to tasks were presented at the blackboard. During the professor's presentation of proofs from the textbook it was sometimes emphasised that certain steps in the proof were the result of a good idea based on a lot of experience, or for instance that a specific delta in an epsilon-delta proof was chosen because the professor knew that this would in fact give the right result.

The student interviews reveal that this norm was established in the classroom. The students all agreed that task solving or proof construction required good ideas or the use of tricks. When trying to construct a proof it is necessary to get an idea to base the proof strategy on. But this idea is related to the statement to be proven. The students have troubles realising this and the professor is not emphasising the connection. So even though the professor might not intend to establish this norm (that the proof strategy is unrelated to the statement), his behaviour supports it.

Some of the norms that the professor tries to establish or that are established unintentionally concern proof construction and proof validation. Analysing the teaching practice from the perspective of proof schemes, as will be carried out in the next section, focuses particularly on these two aspects, which means that some overlapping between the ways of analysing the teaching practice will occur. The reason for including an analysis from the perspective of proof schemes is that this perspective distinguishes between several different proof schemes, and thus provides a more detailed analysis.

## 5.6 Promotion of proof schemes in the teaching practice

The solution processes observed in the pilot and in the main study (not yet presented) reveal that many of the students experience great difficulties constructing

proofs and solving tasks where the solution involves a proof. The students have difficulties understanding the problem and instead try to use superficial strategies such as identification of similarities [Lithner, 2003], use algebraic manipulations without understanding the problem, or try to reach a solution by combining the wordings of theorems. Using the notion of proof schemes to analyse the teaching practice the question is therefore: can the teaching practice account for this behaviour?

The framework of proof schemes is developed to describe students' views and conceptions of what count as personally and publicly convincing arguments (for an elaboration, see page 36). The analysis focuses on which proof schemes the professor's actions in the classroom might support or suppress. Based on this analysis it is neither possible to conclude whether the professor intentionally promotes these schemes nor whether the promoted proof schemes actually are adopted by the students and have an effect on their perceptions of proof.

### 5.6.1 External proof schemes

Even though the framework concerns mathematical proof, some of the aspects described in the framework can also be detected in teaching practices not related to mathematics. For instance, issues concerning authority. In the specific teaching practice, the professor often tries in different ways to reduce his own formal authority and to make the classroom a place where students' opinions and arguments are highly valued. He invites the students to pose questions and to answer questions from other students. The students are expected to say if they have difficulties understanding something, and he does not object to interruptions during a proof review if a student is lost in the line of argumentation. Examples of such non-authoritarian behaviour were seen in the excerpt on page 170. Here, Annie is of course aware of the fact that the professor would prefer that she understood how the statement could be justified using the definition, but she obviously feels that it is okay to show that she is not quite conversant with the material yet, "*It all needs to seep through*" (page 171, line 12).

At other occasions the professor tries to reduce the students' authoritarian behaviour by making it clear that there are many possible ways to present mathematics. The professor explains that there is a distinction between his presentation and that of the textbook, so the students have access to at least two different sources of knowledge. This could diminish their authoritarian proof scheme. Some students are displeased with the professor's non-authoritarian behaviour as the next excerpt shows. The professor is at the end of a proof of the statement that a continuous function on a closed and bounded interval is uniformly continuous (theorem 3.39 in appendix C.2). In the proof a standard trick in epsilon- $N$  proofs is used, where an  $N$  is chosen as the maximum of two other natural numbers,  $N_1$  and  $N_2$  securing the convergence of two introduced sequences,  $\{x_{n_k}\}$  and  $\{y_{n_k}\}$ :



- 20 **Professor:** ...  $x_{n_k}$  converges to  $x$  (writes on the blackboard) and  $y_{n_k}$  con-  
 22 verges to  $x$  which entail that there exists a capital  $N$  or that there  
 24 exists an  $N_1$  such that this distance will become smaller than, and an  
 26  $N_2$  and then we take the largest of the two. There exists an  $N$  such  
 that for all  $k$  larger than or equal to capital  $N$ , the distance from  $x_{n_k}$   
 to  $x$  will be smaller than delta and the distance from  $y_{n_k}$  to  $x$  is smaller  
 than delta. Is it okay that I have taken it in one stroke, here with  $N_1$   
 and  $N_2$  and then taking maximum of  $N_1$  and  $N_2$ , which is  $N$ ? Carrie?
- 28 **Carrie:** But, I think that you should write it out, because I wouldn't, when  
 I sat and looked at it at home, guess that this is what has been done  
 30 there. So in that way I feel that it's nice that everything has been  
 written out, because then I can learn it again when I sit at home and  
 32 read it. If you skip it, it might not get into the brain at all. Then it  
 disappears.
- 34 **Professor:** What you could do, alternatively, was to write down a lemma  
 that you could show once and for all.  $x_n$  converges to  $x$ , and  $y_n$   
 36 converges to  $y$  give that for every epsilon larger than zero there exists  
 an  $N$  in the natural numbers such that for all small  $n$  larger than  
 38 capital  $N$  the distance from  $x_n$  to  $x$  will be smaller than epsilon and  
 the distance from  $y_n$  to  $y$  will be smaller than epsilon. And then you  
 40 could go home and prove that once and for all.
- Carrie:** Well, as long as you are a good example for us, right? That is, if  
 42 you don't expect us to write it, then it's okay that you don't write it,  
 but if you expect us to write it then you must do it too.
- 44 **Professor:** At this point I would expect you to write it, but gradually when  
 we sort of have agreed that, eh, you can show it, when you have shown  
 46 such a lemma, then I would not expect it anymore.
- Carrie:** But in principle then, or in fact it would be healthy that you expect  
 48 it a little bit longer.
- Professor:** Okay, but can I avoid writing it up here? You can go home and  
 solve this exercise (he is referring to showing the lemma).
- Carrie:** It's just because, Michael, sometimes it's very annoying, when you  
 52 sit at home and you have an idea what you want to do, but you don't  
 know how to formulate it and I simply don't know where to look it up  
 54 or where I should..or how I should learn this language unless it's from  
 somebody who can.
- 56 **Professor:** So therefore you need some, you can say, examples which have..?  
**Carrie:** Yes I do.
- 58 **Professor:** Okay, I can follow you on that.
- Carrie:** Because, I am not Bolzano, right? I don't just develop how it should  
 60 be done, right?
- Professor:** No, no. But I think it's suitable to go home and show this lemma.

The professor tries to avoid writing too much on the blackboard and this triggers the discussion between him and Carrie. Carrie reveals that she thinks she needs to copy the professor in order to learn the language and the arguments of mathematics, and through the mastering of the mathematical language she can come to understand the mathematics (deduced from her comments starting on line 28, line 41, and line 51). The professor focuses on content and not on formalism when he suggests that they should go home and try to prove the

lemma he formulates, and he does not give in to Carrie's request for writing down something for her to copy (even though he indicates that he understands her request, "*Okay, I can follow you on that*" (line 58)). Through his responses to Carrie the professor sends the signal that the students are not supposed to imitate his presentation, but that they have to make an independent attempt to acquire mathematical knowledge.

The professor's actions do not support the development of ritual and authoritarian proof schemes (this does not mean that the students do not develop them anyway). It is difficult to find any elements in the teaching practice that promote these two sub-schemes. The case is more complicated when it comes to symbolic proof schemes. When constructing proofs at the blackboard, the professor often explains the symbolism with illustrations, and this could suppress the development of symbolic proof schemes. But as he rarely spends much time discussing the meaning of the theorems, or offers overviews of particular proofs, but instead moves directly to the details of the proofs, this reinforces the development of symbolic proof schemes. The next excerpt illustrates these two opposite tendencies. The professor wants to present the proof of the statement that the limit function of a uniformly convergent sequence of continuous functions is continuous (theorem 7.9 in appendix C.2).

- 62     **Professor:** I had planned to run through the three proofs for the three theo-  
63                   rems about uniform convergence for function sequences, and the first  
64                   theorem says that if a sequence of, of continuous functions converges to  
65                   a function, then the limit function is continuous. (writes down) So if a  
66                   sequence  $f_n$  to  $\mathbb{R}$  ... if a sequence  $f_n$  of continuous functions converges  
67                   uniformly to  $f$  from  $E$  to  $\mathbb{R}$  then  $f$  is continuous. Uniform conver-  
68                   gence, you also write it this way with two arrows (he writes ' $f_n \rightrightarrows f$ ').  
69                   ... (non detectable, 10 words). You could say, in some way this is not  
70                   completely standard, this, so usually people feel obligated to say that  
71                   when we write this we mean uniform. The double arrow.
- 72     **Professor:** What should I do? I have my limit function  $f$ , I have some point  
73                    $x$ , I have  $f_n(x)$ , I want to show that my limit function is continuous at  
74                   this point. That is, I want to find a delta which matches this epsilon  
75                   which I have out here. Here I have drawn my epsilon interval around  
76                    $f(x)$ , and I want to find a delta down here ... such that my graph for  $f$ ,  
77                   it's contained in this box. Here it was okay (refers to the illustration).  
78                   How do I make sure that this is what it actually looks like? ... I can  
79                   say, that I know that all my  $f_n$ 's they approximate  $f$  really well, even  
80                   uniformly, so if I start by choosing a capital  $N$  (writes down) .. so for  
81                   all  $x$  in  $E$  the distance between  $f_n(x)$  and  $f(x)$  .. eh .. for  $n$  larger  
82                   than capital  $N$  have to be smaller than epsilon. I write epsilon here,  
83                   at some point I will change it a little bit. So this means that I can be  
84                   sure that my functions  $f_n$ , for instance  $f_N$ , have a graph epsilon away  
85                   from here, at the most. It can be beneficial with one third of epsilon,  
86                   one third is coming out of the analysis, but you can do it and then  
87                   you can divide by three afterwards. So now we know that it's smaller  
88                   than..that the distance is smaller than one third of epsilon. So here we  
89                   know that we at least are very close. The next you can say is that I

90                    know that my  $f_N$ , it's continuous. So I know, that when I look at the graph for it..it doesn't move much, if I make delta sufficiently small.

The professor finishes the proof without involving the students. In the beginning of the excerpt the professor quickly tells the students the formulation of the statement that he wants to prove, and then he writes down the precise wording (lines 65-67). He does not stop to motivate the statement nor discuss why it is interesting to know that the limit function is continuous. He goes directly to the proof of the statement. He does not try to involve or activate the students in the proof construction. He only poses a rhetorical question, "*What should I do?*" (line 72) and answers it by implicit reference to the definition of continuity (lines 74-75). The professor's validation of the proof assumes that the students have fully acquired the definitions of continuity and uniform convergence of sequences. These definitions are not recollected during the validation process. The professor refers to the analysis or evaluation of the distance  $|f(x) - f(a)|$  (line 86), and uses it as an argument to choose a certain value of delta, and thus assumes that the students already know what this analysis will look like.

It appears to be difficult or demanding for the students to understand the professor's argumentation and that might make the students focus on the symbolic manipulations as the only thing in the proof they can relate to. The lack of a comprehensible explanation of the argumentation might thus lead to the development of symbolic proof schemes. The crucial point is whether the students can comprehend the professor's explanations or not. The fact that the professor illustrates the situation and uses this illustration throughout the validation (e.g. lines 83-85) enhances the students' possibility of understanding the explanations, which suppresses the development of symbolic proof schemes.

### 5.6.2 Empirical proof schemes

Empirical proof schemes concern cases where students believe that statements can be proven by examination of specific instances or by perception. So evidence for the promotion of empirical proof schemes in the teaching practice must be found in situations where specific elements such as specific functions, sequences, sets and so on are used in proofs or in solutions to tasks. If the introduction of specific instances is not accompanied by a thorough explanation it is likely that students do not understand why some proofs using specific elements are valid, while others are not.

In the following excerpt the professor reviews the proof of the sequential characterisation of limits (theorem 3.6), the limit  $L = \lim_{x \rightarrow x_0} f(x)$  exists if and only if  $f(x_n) \rightarrow a$  for  $n \rightarrow x_0$  for every sequence  $x_n$  that converges to  $x_0$  (the proof and the professor's review of it will be analysed in detail in section 5.9). He has reached the second part of the proof, where he must show that for every convergent sequence,  $f(x_n) \rightarrow a$  for  $n \rightarrow x_0$ , the limit of  $f(x)$  for  $x \rightarrow x_0$  exists and is equal to  $a$ . This is proved by contradiction. The excerpt starts where the

professor negates the conclusion,  $f(x)$  converges to  $a$ :

92      **Professor:** This here (points at the statement ' $f(x)$  does not converge to  
                   $a$ ') has to be the same as, there exists at least one epsilon, ..

94      **Carrie:** Yes.

**Professor:** such that no matter what delta I have, for any delta, then there  
 96            is at least one  $x_\delta$  where the function value is further away from  $a$  than  
                  epsilon. That assures, especially, that there does not exist a delta  
 98            because no matter what candidate we have for delta, then there is  
                  at least one  $x$  which says 'doesn't work, go away', even though you  
 100           are as close as delta, then I can get the function values further away  
                  than epsilon (pause). So now I have negated to see if that leads to  
 102           something. We know that every time a sequence converges to  $x_0$  then  
                  the function values converge to  $a$ . Couldn't we find a suitable sequence  
 104           of  $x$ 's here? For instance, you could say (writes down) choose  $\delta_n$  to  
                  be  $\frac{1}{n}$  (looks at the students). This is greater than zero, right? So this  
 106           entails that the distance to  $x_0$  is smaller than delta and such that the  
                  distance from  $f(x_n)$  to  $a$  is larger than epsilon. And that must be  
 108           possible for every  $n$ , right?

**Brian:** What does it say?  $x_n$  minus what?

110      **Professor:** 0, absolute value, has to lie between zero and delta. Now, it says  
                  'choose' here (erases 'choose') you could write ...for every  $n$  in the  
 112           natural numbers, if  $\delta_n$  is  $\frac{1}{n}$ , which is bigger than zero, then there exists  
                  an  $x_n$  which is  $\delta_n$  or  $\frac{1}{n}$  away from  $x_0$  at the most, and with a distance  
 114           to  $a$  which is bigger than epsilon (he looks at the students). Do you  
                  follow, Carrie?

116      **Carrie:** Eh?

**Professor:** The answer is no. What about you, Marie?

118      **Marie:** Yes.

**Professor:** You are able to follow, sort of?

120      **Annie:** Are you looking at me?

**Professor:** It could be you. Are you able to follow?

122      **Annie:** I don't understand, why it has to be  $\frac{1}{n}$ , really.

**Professor:** It could as well have been  $\frac{1}{2n}$  or something else.

124      **Annie:** It was just something that I wondered about when I read it.

**Professor:** Yes.

126      **Annie:** That I didn't understand.

**Professor:** What I can say is, if we look a bit ahead in my agenda, then it  
 128           says, I want to find a sequence  $x_n$  which converges to  $x_0$  and how can  
                  I be sure that a sequence converges to  $x_0$ , I could do that by making  
 130           sure that I squeeze it between something which also goes to zero. That  
                  is, I squeeze  $x_n$  to  $x_0$  so it's smaller than, so the distance is smaller  
 132           than  $\frac{1}{n}$ . Then I am sure that my sequence  $x_n$  converges to  $x_0$ . So that  
                  is really what it's all about, it's about producing a sequence  $x_n$  which  
 134           converges to  $x_0$  and where all the images are epsilon or more away  
                  from  $a$ . And I am simply going to use this  $\delta_n$  to squeeze  $x_n$  down to  
 136            $x_0$  when  $n$  becomes large. So that is why I have delta equal to  $\frac{1}{n}$ . But  
                  an arbitrary sequence  $\delta_n$  tending to zero would be enough. It's just  
 138           because,  $\frac{1}{n}$ , we all know that that one tends to zero.

**Carrie:** Okay,  $\delta_n$  is that now a sequence?

140      **Professor:** It is going to be, yes ... because for every  $n$  there is a new number.

**Carrie:** Okay.

146 **Professor:** Yes (hesitant). But I can do it for every  $n$ , so this is why I can extract a sequence.

Students with a perceptual proof scheme base argumentation on their perception of a problem situation, for instance an illustration of the situation. The professor uses illustrations when reviewing proofs and he expresses the view that

making illustrations is a very powerful tool, but he also emphasises that illustrations can mislead:

**Professor:** You can say that when you have to do something, then you always  
 2        have to make a drawing, and the textbook has a, eh, belongs to a  
        school that says that you can always be seduced by drawings to believe  
 4        something is true because you can draw it. There are things you think  
        you can draw but that you really can't draw. But my experience says  
 6        that you will benefit the most if you let your intuition be guided by  
        drawings.

The professor is aware of the possible pitfalls of using illustrations, and he tries to explain this to the students. Based on classroom observations it is not possible to conclude whether the students understand the possible pitfalls of using illustrations.

### 5.6.3 Deductive proof schemes

The teaching practice observed aims in general at developing students' deductive proof schemes, which is in accordance with the professor's description of the course. The analysis in section 5.3 showed that most of the lesson time is devoted to proof reviews, mostly proofs from the textbook but sometimes the professor also presents proofs in connection with task solving. The teaching practice signals that being able to read and understand the deductive steps in the proofs and to construct proofs are essential competencies that the professor expects the students to develop through participation in the course. He emphasises that it is important to check argumentation and provide valid justification for proposed claims, also when something seems intuitively correct. This is illustrated in the next excerpt where the professor encourages the students to determine if the function  $f : \{0\} \rightarrow \mathbb{R}$ , where  $f(0) = 0$ , is continuous. The professor asks the students:

**Professor:** Is this function continuous at zero?  
 2        (some students laugh, other answer yes, warily.)  
**Professor:** Should we make a vote? Who believes that it's continuous at  
 4        zero? There are at least three. Four, five. Who then believes that it's  
        not continuous? Two, three, four, five.  
 6        **Student:** Then it's a tie. (They laugh)  
**Professor:** Can we examine it? Dennis, you were pretty sure, sort of. What  
 8        did you do to examine the matter?  
**Dennis:** It was more on intuitive ground.  
 10       **Professor:** It was pure intuition. Marie, what did you do?  
**Marie:** I did not do anything. It was just pure ..  
 12       **Professor:** Pure reflex.  
**Tom:** But, I thought about the domain. It is defined in that point and that  
 14       should be enough.  
**Professor:** It's defined in the point, yes, yes, that's right.  
 16       **Tom:** And then according to that one, that definition, then ..

- Professor:** So should we check?
- 18 **Tom:** But it's not defined elsewhere.
- 20 **Professor:** What if we have an epsilon, right? Then we have to find a delta,  
 22 so let's say we have a delta equal to one. Then we have to check if  
 24 it's such that for all  $x$  in  $E$  with the property that the distance from  
 26  $x$  down to  $a$ , is zero smaller than one, then  $f(x)$  minus  $f(a)$  has to  
 28 be smaller than epsilon. Eh, what  $x$ 's do we have? (some say one,  
 30 presumably to indicate that there is only *one* element in  $E$ , others say  
 zero)
- Professor:** We have zero, that is basically the only point in the set we have  
 to work with. For all  $x$ , that is for  $x$  equal to zero, it's the only one,  
 then we (mumbles the rest of the explanation while he points at the  
 blackboard), .. zero smaller than epsilon. So should we say that it's  
 trivially fulfilled?

For fun, the professor suggests that they vote to determine whether the function is continuous or not. It is clear from the students' laughter and responses that this is not an acceptable way to determine the truth of mathematical claims, and the professor indicates quickly that voting is not enough. An examination is needed. Dennis and Marie reveal (line 9 and line 11) that they base their conviction on intuition and reflexes. In the lack of acceptable responses, the professor applies the definition of continuity, although without making this explicit (lines 19-23). By his actions he shows the students that it is necessary to examine and justify claims, and that it is possible to do so by using definitions. This was also identified as an established norm concerning accepted forms of argumentation, see section 5.5.2.

The professor tries to make the students realise that the way the textbook presents mathematical analysis is not unique. Definitions and theorems can be stated differently, and some of the axioms can become theorems if other theorems are set as axioms in stead. Provided that the students understand what the professor is talking about (which is not necessarily the case since they never participate in discussions concerning the nature of mathematics), their axiomatic proof schemes might be strengthened.

It happens several times that the professor makes comments about the quality of a proof by contradiction during a proof review. For instance,

- So how can we show the other implication? The only proofs we really know are those, eh, proofs by contradiction, eh, so this is what I would suggest unless you can come up with something better.
- ..those tools ... we get to work with are proof by contradiction. If you had more tools then you could make a direct proof.
- You can say, we make a proof by contradiction, so it's not really constructive, ...you can't take the proof and solve a task with it afterwards.

It seems that the professor prefers direct proofs – at least in an educational context – since they are constructive and can thus be used as templates for

the students when they try to solve tasks. It is not possible to conclude if the professor's comments might lead the students to distrust proof by contradiction and thereby strengthen their causal proof schemes.

Another way to examine the kind of proof scheme the professor was trying to establish is to analyse the kind of tasks the professor assigned as hand-in tasks.

## 5.7 Categorising hand-in assignments

To characterise further the implemented learning goals, the weekly written hand-in assignments (the task formulation, not the students' written answers) have been analysed. The professor offered and encouraged the students to hand in written answers to selected tasks from the textbook once a week. He would then comment on them. In the student interview the students who handed in the assignments expressed that they valued the comments. Approximately two tasks from each chapter were assigned as hand-in assignments.

Characterising and classifying these assignments gives a picture of the types of tasks the professor expects the students to be able to solve. This is also part of the implemented learning goals. I propose to consider the following five categories:

- **Calculational tasks** where the solver is asked to calculate something. An example: *Evaluate the following limit if it exists:  $\lim_{x \rightarrow \infty} \arctan x$ .*
- **Concept familiarising tasks** which aim at familiarising the student with a concept. An example: *Find the infimum and supremum of the following set:  $E = \{4, 3, 2, 1, 8, 7, 6, 5\}$ .*
- **General proof tasks** where the solver is asked to prove a statement and where none of the involved objects are specific. An example: *Suppose that  $f_n \rightarrow f$  and  $g_n \rightarrow g$ , as  $n \rightarrow \infty$ , uniformly on some set  $E \subset \mathbb{R}$ . Prove that  $f_n g_n \rightarrow fg$  pointwise on  $E$ .*
- **Specific proof tasks** where the solver is asked to prove a statement and the task involves specific functions, sets, sequences, etc. An example: *Let  $E(x) = \sum_{k=0}^{\infty} x^k/k!$ . Prove that the series defining  $E(x)$  converges uniformly on any closed interval  $[a, b]$ .*
- **Prescribed proof tasks** are tasks where the solver is asked to prove a statement, but is told to use a specific statement (definition or theorem) from the chapter. An example: *Using the Inverse Function Theorem, prove that  $(\arcsin x)' = 1/\sqrt{1-x^2}$  for  $x \in (-1, 1)$  and  $(\arctan x)' = 1/(1+x^2)$  for  $x \in (-\infty, \infty)$ .*

The professor appointed 57 tasks as hand-in assignments (written in bold in the course plan, see appendix A). Some tasks have characteristics that place them in two different groups, for example:

- (a) Suppose that  $f$  is improperly integrable on  $[0, \infty)$ . Prove that if  $L = \lim_{x \rightarrow \infty} f(x)$  exists, then  $L = 0$ .



(b) Let

$$f(x) = \begin{cases} 1 & n \leq x < n + 2^{-n}, n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that  $f$  is improperly integrable on  $[0, \infty)$ , but  $\lim_{x \rightarrow \infty} f(x)$  does not exist. [Wade, 2004, p. 142]

The first question is general in nature (thus belonging to the third category), while the second question contains a specific function (belonging to the fourth category). One solution could be to base the characterisation on subtasks instead, but that would distort the picture since tasks belonging to the first and second category in general have more subtasks than tasks belonging to the three other categories. Therefore, the result of the categorisation, table 5.1, both contains the distribution of tasks in the five categories, the distribution of subtasks, and the number of tasks belonging to another category as well (none of the tasks belonged to three or more categories).

Calculational tasks	Concept familiarising tasks	General proof tasks	Specific proof tasks	Prescribed proof tasks
9 (35) [1]	4 (15) [2]	24 (46) [2]	17 (40) [6]	10 (24) [3]

**Table 5.1** Distribution of hand-in tasks in the five identified categories. Within the 57 tasks there were 159 subtasks. The first number indicates the number of tasks in the particular category. The second number, in brackets, indicates the number of subtasks. The third number, in square brackets, indicates the number of tasks belonging to another category as well.

The study of the hand-in tasks suggests that the professor in his choice of tasks views ‘general proof tasks’ as the most important ones, since he assigned most tasks of this kind. This supports the conclusion that the professor attempts to promote deductive proof schemes.

In the student interviews, most of the students indicated that they focused on the hand-in assignments when they solved tasks. They would try to solve them first. The students in the pilot study also paid most attention to the hand-in tasks as many of the students viewed them as an indication of the kinds of tasks the professor expected them to be able to solve at the final exam.

## 5.8 The focus on task solving in the teaching practice

As mentioned, each lesson in the course was primarily divided in a lecture part and a solving session part. Until now, most of the examples characterising the teaching practice have been taken from the lecture part of the lessons, and have

not included activities related to task solving. In this section, I will partly focus on task solving activities in the lectures and partly on the professor-student interaction in the solving sessions.

### 5.8.1 Task solving during the lectures

During the course, I never observed that a student would present a solution to a task at the blackboard in front of the whole class. When solutions were reviewed, the professor would be in charge of the presentation. As mentioned, he made use of an activity where the students would get a period of time to look at a particular task, and then they would discuss it collectively. Often the students would not have completed the task and the professor would have to go through the solution. It seems that the activity enhanced the students' motivation and ability to follow the professor's explanations. He tried on many occasions to involve the students, mostly by asking them about the argumentation behind their choices or to recite formal definitions. The latter was often difficult for the students to do. The following excerpt shows that the students base their strategy choice on comparisons between the task and textbook theorems. The professor tries to make the students provide proper argumentation (the sociomathematical norm of accepted forms of argumentation), and to make the other students evaluate the arguments (social norm of participation).

- Professor:** Okay, Carl. Do you want to give it a shot?
- 32 **Carl:** I don't know. I can start out by making a fool out of myself.
- Professor:** That's totally okay.
- 34 **Carl:** We can see that it resembles..that it's the same formula as the other one, where..
- 36 **Professor:** The other?
- Carl:** That's theorem 5.34. It says  $\phi$  is strictly increasing, that is, ours is
- 38 just strictly monotone. That's the only difference between the two. So far as I could see.
- 40 **Professor:** How did you reach that conclusion?
- Carl:** Because the differential quotient is not zero, so it's either positive or
- 42 negative. So either the function is increasing or decreasing.
- Professor:** What do you say to that? (he addresses the other students) Have
- 44 you reached the same conclusion?

When the students have not looked at the tasks in advance, the presentation of the solutions resembles to a great extent how textbook proofs are validated. As the analysis of the time-line representations showed (see section 5.3.3), not much time was devoted to discussing solution strategies during task solution activities in the lectures. The link between the chosen proof strategy and the statement to be proven was often not addressed explicitly. The following excerpt shows an exception. The professor is solving textbook tasks at the blackboard in front of the class and the students get to choose the tasks he should solve. A student has chosen the task: prove that the series  $\sum_{k=0}^{\infty} 2^k e^{-k}$  converges and find its sum (task 6.1.2b in [Wade, 2004, p. 158]).

- 46       **Professor:** Are there any suggestions? What does it look like? Are there  
any suggestions? I mean, how many convergent series do we know? I  
am just asking. (15 sec. pause)
- 48       **Carl:** (He whispers) 23.
- 50       **Professor:** Guess a theorem, that's what this is all about. How many of you  
think that it looks like a telescoping series? (nobody answers) How  
many thinks it looks like a geometric series? Do we know any other  
52       series? (Carl is laughing) Let me help you a bit.

To know one's mathematical tool box is essential in a task solution situation. It can be a help in deciding on a strategy to use, although it comes down to 'guess a theorem', as the professor calls it. By solving a task like this the students are familiarised with the tools in the tool box, so the task enhances their technical skills and also mathematical resources.

### 5.8.2 Professor-student interaction in the solving sessions

The excerpt on page 171 illustrated that the professor in some situations helped the students by giving them a hint or what they conceived as a trick, which resulted in *guided reasoning* [Lithner, 2008], where the professor would guide them through to a solution.

Other types of professor-student interaction could occur. The following excerpt shows how the professor is able to make the students argue in a more precise and correct way. The situation takes place in the fifth lesson. Two students are trying to solve the following task:

**Task 2.4** (a) Suppose that  $\{b_n\}$  is a sequence of nonnegative numbers that converges to 0, and  $\{x_n\}$  is a real sequence that satisfies  $|x_n - a| \leq b_n$  for large  $n$ . Prove that  $x_n$  converges to  $a$ . [Wade, 2004, p. 38]

The excerpt begins after the two students have worked on the task for twelve minutes. One of the students, Carrie, suddenly sees that they might be able to use the definition of a convergent sequence of real numbers (definition 2.1 in appendix C.1):

- 54       **Carrie:** Well, first I plug in  $b_n$  (into the definition).
- 56       **Iris:** Yes.
- 58       **Carrie:** Okay, you did too, and according to definition 2.1,  $b_n$  minus 0 is  
smaller than epsilon [Iris: Yes] then  $b_n$  is smaller than epsilon. And  
this sequence was smaller than  $b_n$  and  $b_n$  is smaller than epsilon [Iris:  
Yes]. Then this (the sequence  $\{x_n\}$ ) must also be smaller than epsilon.  
And when it's written in this way, then it's a clear example of definition  
2.1 which is exactly the definition of when a sequence converges. . . then  
that's pretty much it, right?
- 62       **Iris:** Yeah.
- 64       **Carrie:** Should we get it checked, whether we're too fast or whether we're  
just..
- Iris:** Really good.

- 66 **Carrie:** Yes.  
 (They raise their hands, and wait for the professor)
- 68 **Carrie:** Well if it's right (their solution) then we're starting to shape up, I think.
- 70 **Iris:** Yes.  
 (They look in the textbook, while they wait for the professor, pause 15 sec.)
- 72 **Carrie:** Which task did we just try to solve, was it number four?
- 74 **Iris:** Mmm.  
 (they wait for 15 sec.)
- 76 **Iris:** You're supposed to... prove that  $x_n$  tends to  $a$ .  
**Carrie:** That  $x$  tends to  $a$ ?
- 78 **Iris:** Yes.  
**Carrie:** Yes.
- 80 **Iris:** Have we done that? (they giggle)
- 82 **Carrie:** Should we check ourselves before we ask? Here ...show that  $x_n$  converges to  $a$ , well I think we do that, because we show that it's smaller than epsilon for a suitable large  $n$ .

Although it takes the two students almost fifteen minutes, they nevertheless manage to reach something very close to a solution, especially taking Carrie's last comment ("*we show that it's smaller than epsilon for a suitable large  $n$* ") into consideration. They give up trying to get the professor to check their solution (because a lot of other groups are fighting for his attention), and instead they proceed to task 2.6:

**Task 2.6** (a) Suppose that  $\{x_n\}$  and  $\{y_n\}$  converge to the same point. Prove that  $x_n - y_n \rightarrow 0$  as  $n \rightarrow \infty$ . [Wade, 2004, p. 38]

After having guessed that they also have to use definition 2.1, but find themselves unable to do so, they manage to get the professor's attention:

- 84 **Professor:** Yes?
- 86 **Carrie:** 6a. But we also want to hear if we're on the right track with this task four, right?
- Iris:** Yes.
- 88 **Carrie:** We just want to hear. Eh.
- Professor:** So what did you do?
- 90 **Carrie:** We used definition 2.1 and first we plugged in  $b_n$ . We found out that  $b_n$  is smaller than epsilon and then we know that this sequence is smaller than  $b_n$  and  $b_n$  is smaller than epsilon and then this sequence has to be smaller than epsilon and then according to definition 2.1, then it must converge to  $a$ .
- 92 **Professor:** That sounds reasonable. I would also have said, given an epsilon, I can find an  $N$  such that  $b_n$  is smaller than epsilon. Yes. But that's what you mean?
- 94 **Carrie:** Yes, that's what we mean, but is it enough? We were sort of wondering.
- 96 **Professor:** Yes, that's what it's all about, you could say. It's about taking the definition of convergence and then say, what does it mean?
- 98
- 100

- 102 **Carrie:** Yes.  
**Professor:** Yes, but that means that given some epsilon, you could find an  
 104  $N$ .  
**Carrie:** Yes, yes.  
 106 **Professor:** And where is epsilon coming from? We want this done,  $x_n$  minus  
 a smaller than epsilon, so there we have got an arbitrary epsilon and  
 108 then we plug it into the definition for the convergence of  $b_n$  and get  
 an  $N$  out, back to the other definition, back to the definition one more  
 110 time with  $x_n$ . So it's exactly..this is the way to do it.  
**Carrie:** Good.

The professor emphasises that they have to include  $N$  in their argumentation. Carrie does not seem to pay attention to this omission in their argumentation. She is more keen on getting his approval than understanding his justification. They move on to ask about task 2.6:

- 112 **Professor:** In task six.  
**Carrie:** There we have not quite reached a good idea. When we browse  
 114 through the chapter, we still think that it's definition 2.1 that's most  
 relevant to use.  
 116 **Professor:** It is.  
**Carrie:** But we're not exactly sure how to juggle with it to get what we  
 118 need.  
**Professor:** Have you tried to make a drawing?  
 120 **Iris:** No.  
**Professor:** Yes, but you just don't know if it's from each side.  
 122 **Carrie:** Nope.  
**Professor:** Try to draw it. Draw a number line with a point  $a$  ... and an  
 124 epsilon on each side.  
**Carrie:** Yes, then it's  $a + \epsilon$  and  $a - \epsilon$ .  
 126 **Professor:** What we basically know is that if we have an  $N$ , no, if we have  
 an epsilon, then we can find an  $N$ , so  $x_n$  is in there and another so  $y_n$   
 128 is in there. Then we take the largest so both of them are in there (in  
 the epsilon-interval around  $a$ ). How big is the difference?  
 130 **Carrie:** Then their difference is smaller than ...?  
**Professor:** Than  $2\epsilon$ !  
 132 **Carrie:** Than  $2\epsilon$ , yes. Yes, but we did work that out algebraically by looking  
 at it.  
 134 **Professor:** What if you..  
**Carrie:** Oh, yes. This is  $2\epsilon$ .  
 136 **Professor:** But what if we want to make it smaller than epsilon what do we  
 do?  
 138 **Carrie:** Either find an  $N$  which is closer..  
**Professor:** Yes, which did what?  
 140 **Carrie:** Which made us come even closer to  $a$ .  
**Professor:** Yes, how close?  
 142 **Carrie:** Very close.  
**Professor:** What about one half of epsilon?  
 144 **Carrie:** Oh, yes, yes. Smaller than one half of epsilon, for instance, well yes,  
 then we use this trick, so we choose an  $N$  so they get smaller than half  
 146 of epsilon and then, okay, in that way. Well, yes.

- Professor:** I don't know if you're following (addresses Iris) or else..?
- 148 **Carrie:** Yes, we'll talk about it. (the professor walks away) Okay, it's still  
2.1. There is just something, we have to remember to use, and that's  
150 this  $N$ .
- Iris:** Yes?
- 152 **Carrie:** Well, here they also say that if you take a really big  $N$  then this is  
also smaller than epsilon [Iris: Yes] and then we can just say that we  
154 choose our  $N$  so big that this is smaller than half of epsilon. Because,  
if we can just make this smaller than half of epsilon and this smaller  
156 than half of epsilon ... then, what's next?

The two students have tried to reach the conclusion, that  $|x_n - y_n| < \epsilon$  only by using symbolic manipulations. The professor tries an approach which aims at providing understanding of the manipulations (line 123). But even with the illustration Carrie still regards it as a trick to choose an  $N$  such that the difference is smaller than half of epsilon (lines 144-146). This is also another piece of evidence that this particular sociomathematical norm (that task solutions demand tricks and good ideas, see section 5.5.3) was established among the students. It also illustrates that steps in a proofs that the professor would not think of as a good idea or a trick, but something that follows logically from the analysis, is regarded by the students as a trick.

When Carrie turns over to explain the line of argumentation to Iris (line 148), she reveals that she has in fact got something out of the professor's comments on their solution to task 2.4. She is now aware of the fact that they need to be more explicit about the choice of  $N$ . After the excerpt stops, the two students try to derive the inequality  $|x_n - y_n| < \epsilon$  by subtracting  $|x_n - a|$  and  $|y_n - a|$  and using their knowledge that  $|x_n - a| < \epsilon/2$  and  $|y_n - a| < \epsilon/2$ . They do not manage to reach a solution, because they start with  $|x_n - a|$  and  $|y_n - a|$  instead of  $|x_n - y_n|$ , and because they only have the triangle inequality present and not the method of adding and subtracting the same number to  $x_n - y_n$ .

The above examples show that professor-student interaction during the solving sessions can have different outcomes. The professor functions as an authority figure, who is there to check that the students have reached a correct solution. Although the students might be uninterested in his more elaborate comments, he manages to make the students realise important shortcomings in their argumentation, and he tries to help the students to enhance their understanding of the symbolic manipulations. What is never explicitly mentioned during the solving sessions (so far as my recordings show) are general strategies of justification. I have never observed that the professor and a group of students discussed the professor's suggested solution strategy in connection with the formulation of the statement to be proven. The discussions between professor and students are confined to the specific task that they discuss the solution of, and the discussion never reaches a more general level. It could be that the students do not acquire general solving strategies that can help them in new solving situations from participating in the solving sessions.

## 5.9 Examining the hypothesis

After having examined general features of the teaching practice using the developed observation template, the notions of social and sociomathematical norms, and proof schemes, it is time to return to the hypothesis based on the theoretical construction of structure, components and details put forward in the previous chapter. In order to either substantiate or refute the hypothesis, it is necessary first to examine whether or not the proof validation situations in the course in the main study can be successfully analysed using the proposed theoretical framework. The second part of the hypothesis concerning the solution processes will be examined in the next chapter, chapter 6. Parts of the following analyses have been presented elsewhere in [Timmermann, 2007a] and [Timmermann, 2007b].

### 5.9.1 Analysing a proof

As an example, a theorem that the reader has already been acquainted with when examining the promotion of external proof schemes (on page 176) is chosen. The precise formulation of the theorem used in the textbook is:

**Theorem 3.6** [Sequential Characterisation of Limits]

Let  $a \in \mathbb{R}$ , let  $I$  be an open interval that contains  $a$ , and let  $f$  be a real function defined everywhere on  $I$  except possibly at  $a$ . Then

$$L = \lim_{x \rightarrow a} f(x)$$

exists if and only if  $f(x_n) \rightarrow L$  as  $n \rightarrow \infty$  for every sequence  $x_n \in I \setminus \{a\}$  that converges to  $a$  as  $n \rightarrow \infty$ . [Wade, 2004, p. 60]

I have included numbers in brackets in the proof, which will be used in the analysis. Beside those numbers, the proof is a verbatim reproduction of the textbook proof. In the proof, the number (1) refers to an implication in the definition of a convergent sequence (definition 3.1):  $0 < |x - a| < \delta$  implies  $|f(x) - L| < \epsilon$ .

#### Proof

[1] Suppose that  $f$  converges to  $L$  as  $x$  approaches  $a$ . Then given  $\epsilon > 0$  there is a  $\delta > 0$  such that (1) holds. [2] If  $x_n \in I \setminus \{a\}$  converges to  $a$  as  $x_n \rightarrow \infty$ , then choose an  $N \in \mathbb{N}$  such that  $n > N$  implies  $|x_n - a| < \delta$ . [3] Since  $x_n \neq a$ , [4] it follows from (1) that  $|f(x_n) - L| < \epsilon$  for all  $n > N$ . Therefore,  $f(x_n) \rightarrow L$  as  $n \rightarrow \infty$ . [5] Conversely, suppose that  $f(x_n) \rightarrow L$  as  $n \rightarrow \infty$  for every sequence  $x_n \in I \setminus \{a\}$  that converges to  $a$ . [6] If  $f$  does not converge to  $L$  as  $x$  approaches  $a$ , then there is an  $\epsilon > 0$  (call it  $\epsilon_0$ ) such that the implication ' $0 < |x - a| < \delta$  implies  $|f(x) - L| < \epsilon_0$ ' does not hold for any  $\delta > 0$ . [7] Thus, for each  $\delta = 1/n$ ,  $n \in \mathbb{N}$  there is a point  $x_n \in I$  that satisfies two conditions:  $0 < |x_n - a| < 1/n$  and  $|f(x_n) - L| \geq \epsilon_0$ . [8] Now the first condition and the Squeeze Theorem (Theorem 2.9) imply that  $x_n > a$  and  $x_n \rightarrow a$ , so by hypothesis,  $f(x_n) \rightarrow L$  as  $n \rightarrow \infty$ . In particular,  $|f(x_n) - L| < \epsilon_0$  for  $n$  large, which contradicts the second condition. [Wade, 2004, p. 60]  $\square$

Theorem 3.6 includes a bi-implication, an ‘if and only if’-sentence, and the majority of proofs of such theorems are structured in two parts where the implications are shown separately. The chosen strategy is to prove the first implication ‘ $\Rightarrow$ ’ with a direct proof whereas the second implication ‘ $\Leftarrow$ ’ is proved indirectly by contradiction. To make a strategy choice, or to understand why a given strategy choice has been made, is an important part of a strategy discussion. In the textbook this strategy choice is not emphasised or discussed.

The theorem has a twist because there is a double hypothesis part. There are thus two premises and one conclusion in the first part:

$P_1 : f(x) \rightarrow L \text{ as } x \rightarrow a.$

$P_2 : x_n \rightarrow a \text{ as } n \rightarrow \infty.$

$Q : f(x_n) \rightarrow L \text{ for } n \rightarrow \infty.$

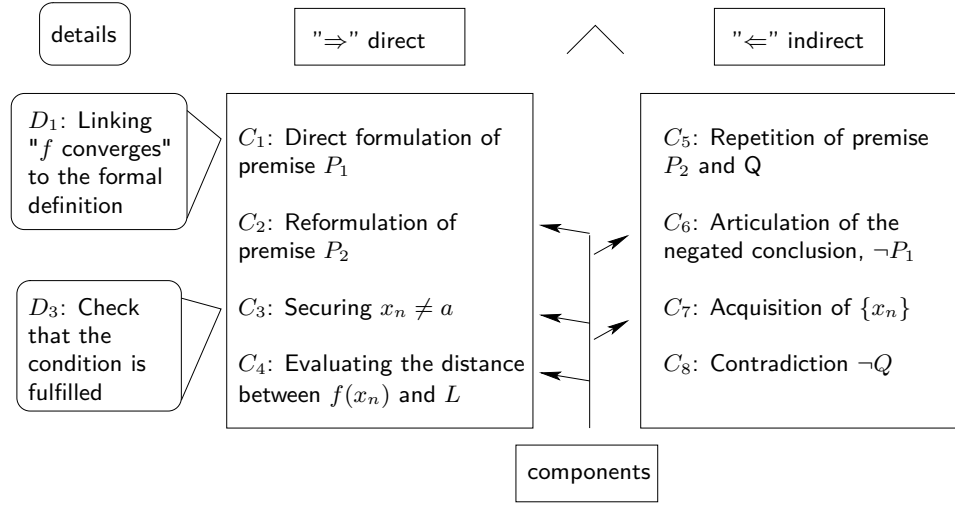
The proof strategy of the first implication thus is: ‘if  $P_1$  and  $P_2$ , then  $Q$ ’, i.e.  $(P_1 \wedge P_2) \Rightarrow Q$ . In the first step the premise  $P_1$  is formulated directly, and premise  $P_2$  is reformulated in the second step. The two steps might at first sight look similar, but the second step deviates from a mere formulation of the premise. It draws the consequences of premise  $P_2$  and is, in that sense, a reformulation of  $P_2$ . The third step provides the missing link before the results obtained so far can be combined; namely securing that  $x_n \neq a$ . In the fourth step, the combination of the formulation of premise  $P_1$ , the reformulation of premise  $P_2$  and the securing leads to the conclusion that  $f(x_n)$  converges to  $L$ . The structure of the proof with the described components is shown in figure 5.6.

In the second part of the proof an indirect proof strategy, proof by contradiction, is chosen for non-explicit reasons.  $P_2$  is still a premise, but now  $Q$  is a premise and  $P_1$  is the conclusion. The logical structure of this part is based on the logical tautology  $[(P_2 \Rightarrow Q) \wedge \neg P_1] \Rightarrow \neg Q \Rightarrow (Q \Rightarrow P_1)$ .

What does it take to realise the different components? What are the details? I will provide some examples. In the first component the formulation of premise  $P_1$  demands a reproduction of the definition of the limit of a function, which includes a repetition of the definition and an ability to switch between the different formulations, phrases and notations used to describe limits of functions. The details of the third component, securing that  $x_n \neq a$ , is just a contemplation that this condition is fulfilled.

The second part of the proof is more complicated in the sense that the details of the four components in this part are more extensive. The proof strategy is proof by contradiction so the sixth step, articulation of the negated conclusion, involves negation of an expression containing multiple quantifiers. In the seventh step, acquisition of  $\{x_n\}$ , a sequence has to be constructed in order to provide the contradiction. The sequence  $\{x_n\}$  remains unspecified, but is defined through the chosen sequence of deltas,  $\delta = 1/n$ , and the negated conclusion. This  $\delta$ -sequence





**Figure 5.6** The structure of the proof is composed of the main steps or components (denoted by the letter C) that the chosen proof strategy leads to. The realisations of the components are the details (denoted by the letter D) of the proof. The details are shown for two of the components as an illustration. The notation used is:  $P_1$ :  $f(x) \rightarrow L$  as  $x \rightarrow a$ ;  $P_2$ :  $x_n \rightarrow a$  as  $n \rightarrow \infty$ , and  $Q$ :  $f(x_n) \rightarrow L$  for  $n \rightarrow \infty$ .

entails that  $\{x_n\}$  converges to  $a$  and that each element  $x_n$  in the sequence  $\{x_n\}$  satisfies the condition  $|f(x_n) - L| \geq \epsilon_0$ .

### 5.9.2 Analysing a proof validation situation

During the presentation of the result/theorem, the professor writes the following formulation of the result on the blackboard and refers to it during his proof review (the professor uses a different notation than the textbook,  $a$  instead of  $L$  and  $x_0$  instead of  $a$ ):

$$f(x) \xrightarrow{x \rightarrow x_0} a \quad \Leftrightarrow \quad \forall \{x_n\}_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}, x_n \xrightarrow{n \rightarrow \infty} x_0 \quad (5.1)$$

$$\Rightarrow f(x_n) \xrightarrow{n \rightarrow \infty} a$$

It takes the professor 25 minutes to go through the proof. He starts with the claim that the first implication is almost trivial. He says that since they have to talk about all sequences they need to pick an arbitrary convergent sequence and see what they can say about that one. Then he proceeds to make a graphical illustration of the situation and the excerpt begins where he comments on his illustration:

$C_1$		
2	<b>Professor:</b> We have a graph $f$ . We have an epsilon-window. We have a matching delta. ... and we have a sequence, eh, $x_n$ converging down to $x_0$ and we want to show that the function values of the sequence converge to $a$ , right? And what does it mean that the sequence converges to $x_0$ ? ... well, then it has to stick to this interval, minus delta to delta, as long as $n$ is big enough. Annie, doesn't it?	$C_2$ $C_4$ $D_2$
6	<b>Annie:</b> I was just gone there for a moment ..	
8	<b>Professor:</b> You were just, yes, okay. We want to show that the sequence of function values $f(x_n)$ converges to $a$ and what we know is that if $x$ is in the delta-interval around $x_0$ , then all the function values are in the epsilon-interval around $a$ . And then I say, if we are to make sure that $f(x_n)$ is at most epsilon away from $a$ then it's basically enough to capture $x_n$ in this interval from minus delta to delta because then we know that the function values are in the right interval ... and there .. Can we make sure that $x_n$ is in the interval from $x_0$ -delta to $x_0$ +delta?	$C_4$ $C_1$ $Q'$ $C_2$
16	<b>Carrie:</b> Has it something to do with choosing an $n$ that is big enough?	$D_2$
18	<b>Professor:</b> That sounds like a really good idea. Can we do that?	$C_2$
18	<b>Carrie:</b> We can do that.	
20	<b>Professor:</b> We can do that. What, eh, how big does it have to be?	$D_2$
20	<b>Tom:</b> Bigger than capital $N$ .	
22	<b>Carrie:</b> Yes, it has to be bigger than capital $N$ .	
22	<b>Professor:</b> No, it's capital $N$ that we are about to choose, right? How big are we going to choose capital $N$ ?	$D_2$ $D_2$
24	<b>Bob:</b> So big, that the difference between the sequence and the limit is smaller than, absolute value, smaller than delta.	
26	<b>Professor:</b> Than delta.	

In the excerpt the professor hastily goes through the first two components (lines 1-3). Then he jumps to the component of the conclusion  $Q$  (“*we want to show ...*”) and finally back to the details of the second component (lines 4-6). After Annie’s sign of lack of attention (line 7) what is the professor then doing? He begins ‘backwards’, starting with conclusion  $Q$  (lines 8-9) which is followed by the first component, ‘formulation of premise  $P_1$ ’ (“*and what we know ... epsilon-interval around a*”). Then he tacitly reformulates the logical structure of the proof (“*if we are to make sure ... minus delta to delta*”): ‘if  $Q$  needs to be true, then it is enough if  $P_2$  is true’. Instead of talking about the necessary condition for  $Q$  to be true (‘if  $P_1$  and  $P_2$ , then  $Q$ ’), he now focuses on a sufficient condition and that draws attention to premise  $P_2$  instead of conclusion  $Q$ . It is presumably difficult for a student to follow this equivalent reformulation when the professor does not explicate what he is doing.

The professor involves the students on five occasions (line 6, 15, 17, 19, and 22). On two of those occasions (line 15 and 17) he poses questions where proper answers would refer to the second component, ‘reformulation of premise  $P_2$ ’: “*yes, because  $x_n$  is chosen to be a convergent sequence*”. The first reply from Carrie refers instead to the details of this component and in her second reply she does not justify her answer. Since Carrie’s reply concerns details, the professor is led to focus on details as well in his following question. On the three other

occasions (line 6, line 19, and line 22), the professor asks with reference to the details of the second component and this is also the response he gets from the students.

Aside from tacitly reformulating the logical structure of the proof the professor also shifts between a components perspective and a details perspective, while the students maintain a focus on the details.

The professor writes down the details of the first two steps. The following excerpt concerns the third component. The details of this step only include an inspection which explains why the professor characterises this step as ‘free’:

- D*<sub>3</sub>
- 34       **Professor:** And then I quickly just want to add, that zero is smaller than  
the distance from  $x_n$  to  $x_0$  and that is because my sequence will never  
reach the value  $x_0$ , right? That is just for free.
- 36       **Carrie:** That is just for free?
- 38       **Professor:** Yes, that is, it’s just there, my sequence was contained in  $I$  with-  
out  $x_0$ , so none of the  $x_n$ ’s can be  $x_0$ .
- 40       **Carrie:** Why is that free?
- 40       **Professor:** Well, I mean, that assures me that the distance is bigger than  
zero. That’s what’s free. When I have paid the other price first, right?

Supposedly, Carrie does not realise the details of this component because the structure of the proof is not clear to her and she does not recognise what role the component plays in the structure. Her uncertainty about the structure and on which part of the structure the discussion is focused makes it impossible for her to comprehend the details of this component.

The professor finishes the first part of the proof, and they have a discussion about how the notation deviates from that of the textbook. The professor moves on to the second part of the proof where he proclaims that he wants to use proof by contradiction if none of the students have any other suggestions. So in his presentation of the proof no emphasis is put on the justification of the strategy choice in neither the first nor the second part of the proof.

- D*<sub>6</sub>
- 86       **Professor:** So now we know that no matter what sequence converging down  
to  $x_0$  the sequence of function values converges to  $a$ . And we want to  
88       conclude that  $f(x)$  converges to  $a$ .  
(6 sec. pause, the professor looks at the students)
- 90       **Professor:** Let’s try to make a contradiction. Let’s assume that  $f(x)$  does  
not converge to  $a$ . (he writes on the blackboard:  $f(x) \nrightarrow a$ ) What does  
92       it mean that  $f(x)$  does not converge to  $a$ ?
- 94       **Bob:** It diverges.
- 94       **Student:** Or it converges to something else.
- 96       **Carrie:** Or it doesn’t approach anything.
- 96       **Professor:** Yes, but can we formulate that as something. Aaron?
- 98       **Aaron:** That the distance between them is larger than epsilon.
- 98       **Professor:** For all  $x$  or ..?
- 98       **Aaron:** For  $x$  approaching  $x_0$ ?

100 **Professor:** So if there had not been this line across then it meant that for  
every epsilon there exists a delta, right? What is the negation of for  
102 every epsilon there exists a delta?

**Carrie:** There doesn't exist a delta.

104 **Bob:** For epsilon there exists ..

**Professor:** Yes, there exists at least one epsilon where there doesn't exist a  
106 delta, right?

**Carrie:** Yes.

108 **Professor:** This means (writes down) assume that there exists an epsilon  
larger than zero, eh, such that for every delta larger than zero there  
110 exists, there is, we could call it  $x_\delta$  where the distance to  $x_0$  is between  
zero and delta and such that the distance from  $f(x_\delta)$  to  $a$  is larger  
112 than epsilon.

**Carrie:** Do that again. There exists ...?

114 **Professor:** This here (points at the statement ' $f(x)$  does not converge to  
 $a$ ') has to be the same as, there exists at least one epsilon, ..

116 **Carrie:** Yes.

**Professor:** such that no matter what delta I have, for any delta, then there  
118 is at least one  $x_\delta$  where the function values are further away from  $a$   
than epsilon. That assures, especially, that there does not exist a delta  
120 because no matter what candidate we have for delta, then there is  
at least one  $x$  which says 'doesn't work, go away', even though you  
122 are as close as delta, then I can get the function values further away  
than epsilon (pause). So now I have negated to see if that leads to  
124 something.

After repeating premise  $Q$  and conclusion  $P_1$  (lines 86-88), the professor guides the students through the details of the component 'articulation of the negated conclusion' (rest of the excerpt). Here both the professor and the participating students are talking about and referring to the details and it is clear from the dialogue that the students are able to follow the professor's guiding, although they find it difficult to provide a formally correct formulation. A reason for the students' ability to follow the guidance might be that the students recognise the link between the strategy choice and the negation component and thus are able to understand the explanation of the details.

After guiding the students through the details of the negation component the professor continues to the seventh step, the 'acquisition component', which leads to difficulties for the students. In the following excerpt, which the reader is already acquainted with, he begins by combining premise  $P_2$  and premise  $Q$  (lines 125-126):

**Professor:** We know that every time the sequence converges to  $x_0$  then the  
126 function values converge to  $a$ . Couldn't we find a suitable sequence of  
 $x$ 's here? For instance, you could say (writes down) choose  $\delta_n$  to be  $\frac{1}{n}$   
128 (looks at the students). This is greater than zero, right? So this entails  
that the distance to  $x_0$  is smaller than delta and such that the distance  
130 from  $f(x_n)$  to  $a$  is larger than epsilon. And that must be possible for  
every  $n$ , right?

132 **Brian:** What does it say?  $x_n$  minus what?

- 134 **Professor:** 0, absolute value, has to lie between zero and delta. Now, it says  
 136 ‘choose’ here (erases ‘choose’) you could write ... for every  $n$  in the  
 138 natural numbers, if  $\delta_n$  is  $\frac{1}{n}$ , which is bigger than zero, then there exists  
 an  $x_n$  which is  $\delta_n$  or  $\frac{1}{n}$  away from  $x_0$  at the most, and with a distance  
 to  $a$  which is bigger than epsilon (he looks at the students). Do you  
 follow, Carrie?
- Carrie:** Eh?.
- 140 **Professor:** The answer is no. What about you, Marie?
- Marie:** Yes.
- 142 **Professor:** You are able to follow, sort of?
- Annie:** Are you looking at me?
- 144 **Professor:** It could be you. Are you able to follow?
- Annie:** I don’t understand, why it has to be  $\frac{1}{n}$ , really.
- 146 **Professor:** It could as well have been  $\frac{1}{2^n}$  or something else.
- Annie:** It was just something that I wondered about when I read it.
- 148 **Professor:** Yes.
- Annie:** That I didn’t understand.
- 150 **Professor:** What I can say is, if we look a bit ahead in my agenda, then it  
 152 says, I want to find a sequence  $x_n$  which converges to  $x_0$  and how can  
 I be sure that a sequence converges to  $x_0$ , I could do that by making  
 sure that I squeeze it between something which also goes to zero. That  
 154 is, I squeeze  $x_n$  to  $x_0$  so it’s smaller than, so the distance is smaller  
 than  $\frac{1}{n}$ . Then I am sure that my sequence  $x_n$  converges to  $x_0$ . So  
 156 that is really what it is all about, it’s about producing a sequence  $x_n$   
 which converges to  $x_0$  and where all the images are epsilon or more  
 158 away from  $a$ . And I am simply going to use this  $\delta_n$  to squeeze  $x_n$  down  
 to  $x_0$  when  $n$  become large. So that is why I have delta equal to  $\frac{1}{n}$ .  
 160 But an arbitrary sequence  $\delta_n$  going to zero would be enough. It’s just  
 because,  $\frac{1}{n}$ , we all know that that one goes to zero.
- 162 **Carrie:** Okay,  $\delta_n$  is that now a sequence?
- Professor:** It is going to be, yes ... because for every  $n$  there is a new number.
- 164 **Carrie:** Okay.
- Professor:** But you have to like say, you can do it individually. Every time  
 166 I take an  $n$ , right, then I can find an  $x$  here.
- Carrie:** Yes, so it’s to show that it’s for every?
- 168 **Professor:** Yes (hesitant). But I can do it for every  $n$ , so this is why I can  
 extract a sequence.
- 170 **Carrie:** Yes.

D<sub>7</sub>

Annie expresses difficulties with the choice of the sequence  $\{1/n\}$ , and the professor tries to explain it while maintaining a focus on the details (lines 150-161). He emphasises the conditions the sequence should fulfil, but he does not mention why they need a specific sequence with these properties in the first place. He thus concentrates on the details of the seventh component, but does not explain how the component relates to the choice of proof strategy.

- 172 **Dennis:** But if  $x_n$  gets arbitrary close to  $x_0$  won’t it be smaller than epsilon?  
 So where ..?

$C_6$  174 **Professor:** Damn good argument. Something is wrong here, right? That is  
 $C_7$  176 the contradiction, right? So we started by assuming that  $f(x)$  did not  
 178 approach  $a$  and then, God help me, then we can produce a sequence  
 which stays far away from  $a$ , but we know by assumption that every  $D_8$   
 time we took a sequence like that, which converges down to  $x_0$  the  
 sequence of function values converges to  $a$ . Something is wrong, right?  
 That is a contradiction. So you are completely right.  $C_8$   
 180 **Student:** Try to say that again.

The professor interprets the question from Dennis as a formulation of the contradiction component, and the professor quickly summarises the components of the second part of the proof.

182 **Carrie:** Yes, but, eh, I might not be completely with you, eh, but the last  
 thing that we used as an argument that it couldn't be, wasn't that what  
 we were supposed to show? Or is it I who have switched something  
 184 around?  
**Professor:** We tried to show, we were about to show this arrow, this way  
 186 (points at ' $\Leftarrow$ ' in the implication (5.1) on page 190), [Carrie: yes],  
 that is, we assume that this is true (points to the right hand side of  
 188 the implication) and then we show that this over here is true (points  
 to the left hand side of the implication). [Carrie: yes] Then we are  
 190 allowed to use what we assume is true.  
**Carrie:** Okay. Yes.  
 192 **Professor:** The thing that we question is not this here (points to the right  
 hand side of the implication), but if this here implies this over here  
 194 (points to the left hand side of the implication).  
**Carrie:** Okay. Sort of. Yes, I actually follow.  
 196 **Professor:** Yes (does not sound completely convinced). You could say, this  
 is always what is important, when you have to show theorems like this  
 198 [Carrie: yes] and some implications, that is, something that implies  
 something else, if it's true then the other thing has to be true. This  
 200 does not mean that this is true [Carrie: No, no] it just sometimes is.  
**Carrie:** But if we assume this is true then it has to imply the other thing.  
 202 **Professor:** Yes.

Carrie is confused about the structure of the proof (lines 181-184). This leads to a clarification of the logical structure of the second part of the proof (lines 185-190) and of the logical structure of a proof of an arbitrary 'if-then' statement (lines 196-202).

### 5.9.3 Results of the analysis

Carrying out an analysis of the proof validation situation based on structure, components and details, provides an interpretation of the professor's explanations, for instance why he describes the first implication as almost trivial (the realisation of the component does almost only imply easy links or straight forward inspections) and why they get the condition  $x_n \neq x_0$  'for free'. The analysis also shows that the professor pays much attention to details, and less attention to components

and thus to the structure of the proof, and when he mentions the components it is not done in the ‘natural order’ (as defined by the textbook). Once, he even reformulates the logical structure of the proof, without making it explicit. He takes for granted that the students are already familiar with the proof structure, and his actions indicate that he might think, that since the structure is clear it is just a matter of filling in the details. The analysis shows that the students focus on details, and at times even direct the professor away from the structure and the components.

There are periods where the students have difficulties following the professor’s explanations, both when the explanations concern easy steps, such as the third step (securing), as well as the more difficult steps, such as the seventh step (acquisition). The students also show abilities to follow the professor and participate in the validation process, also at difficult steps, such as the sixth step (negation). An explanation for these observations could be that the students find it difficult to understand the professor’s detailed explanations when they do not comprehend the location of the particular component in the structure. So although the details of the sixth component (negation) are complicated, the relation between the component and the proof strategy (the relation between negation and proof by contradiction) is immediately recognisable and explained by the professor to the students’ satisfaction.

## 5.10 Results from the supplementary study

The aim of this section is to document that other ways to conduct teaching at university level exist, to show that professors can have different intentions or goals with teaching, and to show that the analysis tool developed can be used to analyse and characterise other teaching practices. For more specific details about the course in the supplementary study, the reader is referred to section 1.4.4 in the introduction.

### 5.10.1 Professor’s intentions

In a written introduction to the course the professor in charge emphasises that the course focuses specifically on both understanding proofs and the construction of proofs:

This course is designed to bridge the gap between intuitive calculus and rigorous mathematical analysis, the place where future mathematics majors learn how to come up with and write detailed proofs. The main objective of this course is to provide a deep understanding of the analysis of the functions of one variable. By the end of the course you will:

- Know and be able to efficiently use the basic principles and methods of logical proofs;
- Know and be able to prove the main theorems on continuous and differentiable functions on the real line;

- Know and be able to prove the basic properties of sequences and series of numbers and sequences and series of functions.

(Professor's written introduction)

The course thus extensively focuses on developing students' reasoning competency (as defined in [Niss et al., 2002]). Based on the three aforementioned goals, it is not obvious to what degree the students should be able to construct new theorems or reproduce known proofs. It is certainly very ambitious if the professor expects the students to be able to prove the main theorems on continuous and differential functions by themselves. So "*being able to prove*" might mean 'being able to reproduce and explain a known proof'.

In an interview with the professor conducted at the end of the course, the professor verifies this interpretation. The aim of his teaching practice is to teach the students the relevant definitions, make them able to reproduce theorems and to provide examples illustrating important properties. He believes that the lectures function as a supplement to the textbook where the students can come to realise the axiomatic structure of mathematical analysis, and also experience that mathematical analysis is fun and useful.

Several different types of assessment are used in student grading. The professor has designed online assignments (based on multiple choice and proofs with missing words) to be answered before every lesson. According to the professor, the multiple choice questions function as a disciplinary tool to get the students to read the textbook, while the proofs with missing words also provide the students with a sense of the structure of a proof, which is meant to help them when they have to construct proof themselves. Traditional pen and paper assignments provide students with the experience of being "*placed in front of a sheet of blank white paper where they are asked to construct something on their own*" (the professor). The first midterm test, which only includes reproductions of definitions and proofs, is designed to send the message that they cannot make their own definitions of the concepts, that they have to be precise, and that they have to learn to reproduce the proofs the professor have shown them in the lectures. Although the purpose of the second midterm test is the same, it also includes one unfamiliar statement to be proven. The final exam includes two unknown statements to be proven. The objective is for students who conscientiously study the textbook to be able to pass the course, while students who are also able to construct proofs of unfamiliar statements can achieve a higher score.

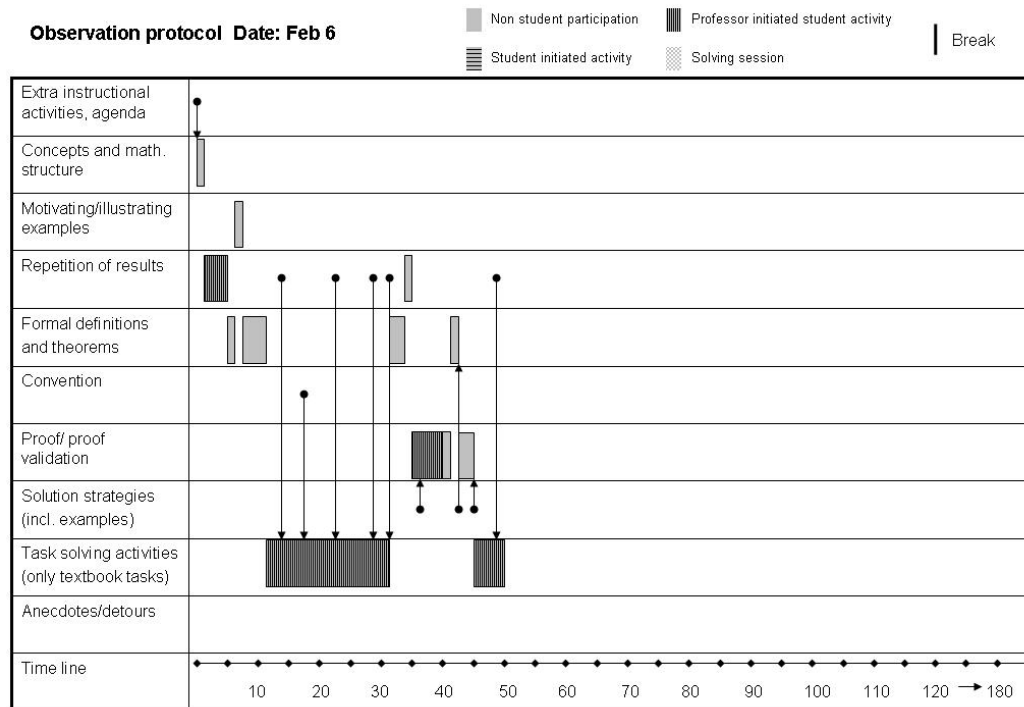
### 5.10.2 Characterisation of teaching practice

Two lessons have been analysed using the analysis tool developed and both time-line representations are listed in appendix B.2. Figure 5.7 shows the time-line representation of one of the lessons. Even though the lessons in this course have a duration of fifty minutes<sup>3</sup>, I have used the same template as in the main study

<sup>3</sup> Interrupted by a fire drill, the other lesson analysed lasted only forty minutes.



which makes it easier to compare lessons from the two courses. The lesson looked at in the following concerns the introduction of topology and central topological notions such as open sets and neighbourhoods.



**Figure 5.7** A time-line representation of a 50-minute lesson. Each period has been characterised according to the ten categories, and the four types of student/professor activity (see page 152). If the duration of an identified period is less than 1.25 minutes, the period is represented by an arrow. Isolated questions from the professor and the students are depicted by full or dotted vertical lines, respectively.

The professor begins the lesson by briefly describing its purpose and by explaining that topology is a notion useful for understanding continuity better (0-1.25 min). He proceeds by recapitulating the notion of a Euclidean distance between two points (1.25-5 min). He invites students to answer his questions. He continues with definitions of a neighbourhood, a deleted neighbourhood, interior points and boundary points (5-11.25 min), including an illustrative example (6.25-7.5 min). The definitions are listed in appendix C.1 (number 13.1, 13.2, and 13.3). Next, he goes through two examples/tasks from his notes<sup>4</sup>: Find interior points and boundary points for the sets  $\{1, 2, 3\}$  and  $[1, 2) \cup (3, 4]$ . As indicated by the length of the time span (11.25-31.25 min), the professor is thorough in

<sup>4</sup> For each lesson the professor makes a handout stating his lesson plan that leaves space for the students to include their own notes.

his presentation and several times he involves the students in the argumentation process. He also spends time referring to the definitions he presented at the beginning of the lesson (the arrows from ‘repetition of results’). At one point he explains that epsilon is always assumed to be very small. This comment is characterised as being one about conventions (the arrow from ‘convention’). After the task solving period, the professor presents the definition of closed and open sets (31.25-33.75 min) (see appendix C.1 number 13.6) and provides an illustrative example (33.75-35 min) before moving on to prove the statement  $S \text{ open} \Leftrightarrow S = \text{int}S$  (35-41.25 min). His proof review will be analysed in more details below. The professor presents another theorem (41.25-42.5 min) and proves it (42.5-45 min) without involving the students. The lesson ends with two tasks (45-50 min) and the students are reminded of the definition of closed sets (arrow).

When validating proofs, the professor often asks the students specific questions. The following excerpt, which contains the proof of the statement ‘ $S$  open iff  $S = \text{int}S$ ’, illustrates the type of questions the professor poses. The symbols in the margin refer to the analysis presented in section 5.10.3.

	<b>Professor:</b> Let me show you this theorem. So ‘proof’, first note that $S$ is a	$C_4$
2	subset..that the interior of $S$ is a subset of $S$ . We are going to prove	
	a set is open if $S$ equals the interior of $S$ . You know, how do we	Structure
4	understand this $i$ double $f$ (‘iff’)?	
	<b>Student:</b> If and only if.	
6	<b>Professor:</b> If and only if. And that means what?	
	<b>Student:</b> Two-way stream.	
8	<b>Professor:</b> Two-way stream. We can go in the direction $S$ is open then $S$	
	equals the interior of $S$ and ... (he might gesticulate the other direc-	
10	tion). Let $S$ be open. What is our definition of open? What is our	$D_2$
	definition of open? What is our definition of open?	
12	<b>Student:</b> That the complement contains the boundary.	
	<b>Professor:</b> So boundary of $S$ , then by definition, boundary of $S$ is a subset	$D_2$
14	of $\mathbb{R}$ without $S$ . What I, what I wanna prove, I wanna prove that here	
	I have equal. I know this, this is my definition of interior point, so I	$D_3$
16	need to go in this direction, I need to show that if I take something	$D_1$
	from $S$ , that that belongs to the interior. So let $x$ belong to $S$ . I would	$C_1$
18	like to conclude that this $x$ belongs to the interior. Since, boundary	
	of $S$ equals $\mathbb{R} \setminus S$ , $x$ doesn’t belong to the boundary. How can I	$D_2, C_2$
20	tell? How can I tell that $x$ doesn’t belong to the boundary of $S$ ?	
	When I said $x$ belongs to $S$ and I know the boundary of $S$ is in the	$C_1$
22	complement, so $x$ doesn’t belong to the boundary. This means that	$D_2$
	there is a neighbourhood $N_\epsilon$ of $x$ such that $N_\epsilon \cap S$ is empty or $N_\epsilon \cap \mathbb{R} \setminus S$	$D_3$
24	is empty. ... Because the definition of boundary is that both of those	
	are non-empty, if it belongs to the boundary both of them are non-	
26	empty. Is this possible? Is this possible? (he probably points at	
	$N_\epsilon \cap S = \emptyset$ )	
28	<b>Student:</b> No.	
	<b>Professor:</b> No, why not? I know that something is there, $x$ is there. Since	$D_3$
30	$x$ belongs to $N_\epsilon \cap S$ , $N_\epsilon \cap S$ is not empty and $N_\epsilon \cap \mathbb{R} \setminus S$ is empty.	
	Therefore, $N_\epsilon$ is a subset of $S$ and $x$ is by definition ... $x$ belongs to	
32	interior of $S$ therefore $S$ is a subset of interior of $S$ . So if open then	$C_3, C_4$

- 34  $S$  equals the interior of  $S$ . Any questions? Any questions? So if  $S$  is open then  $S$  equals the interior of  $S$ .
- $C_5$  **Professor:** Now let  $S = \text{int}S$ . If  $x$  belongs to the boundary of  $S$ , then  $x$
- $C_6$  36 does not belong to the interior of  $S$ . Equals  $S$ . So  $x$  belongs to  $\mathbb{R} \setminus S$ .
- $D_7, C_7$   $S$  is open. And again by definition,  $S$  is open.

The professor eagerly tries to involve the students in interpreting the statement and in the justification of the individual steps in the proof, and he manages to get the students to respond (line 5, line 7, line 12, and line 28), but only with short answers. When the professor asks for an explanation of ‘iff’, a student provides a mere translation first and only when the professor repeats his question does the student provide a very short explanation (“*two-way stream*”), which only makes sense if one already knows what ‘iff’ means. The second time the professor asks the students, he asks explicitly for a definition and a student is able to provide one. The third time the professor addresses the students, only one student replies, giving the answer “*no*”; the professor has to provide the justification.

In the interview, the professor states that by asking questions during proof validations he wants students to experience that they are able to justify particular steps in the proofs independently, and he feels that this objective is met:

What I want to show them are the steps in the proof, and with the questions I want to show them that each of those steps has significance and really leads to some conclusion that we would like to reach. So, and with the questions, I also want to show them that with those small steps I want to make them feel that they can take them on their own. Maybe they don’t see the big picture of the complete proof at once, but that they can do step-by-step. So what I want to say in class is that nobody is born with this knowledge, and really that’s what I am trying to show them, that they can do it even though they don’t see how right now . . . and most of the time I get the answer that I am looking for in the class, so that is also a way to show the rest of the class that it is possible for some of the kids. (Professor, supplementary study)

Both the interview with the professor and the observations clearly show that the professor undertakes a disciplinary role. He plans and orchestrates the lectures, and the students have no influence on how the teaching is carried out, what they talk about during lectures, or the pace of the teaching. The students are allowed to pose questions, but the professor gives the impression that he prefers that they only answer his questions. A lack of time might be a reason for this preference. Since each lesson only lasts fifty minutes, and the professor has a full lesson plan, there is not much time left over for answering questions. Another reason could be that only the brightest students ask questions and that their questions are at a more advanced level than the professor finds suitable, so he often answers them quickly or postpones replying (and only seldom returns to them).

### 5.10.3 Analysis from the perspective of structure, components, and details

What does an analysis of the above proof validation situation show? The theorem appears immediately obvious, so it is likely that the professor has not chosen to present this proof in order to explain the meaning of the statement. It seems more likely that the professor wants to make the students feel that they are able to provide the details of the proof themselves (just as he explains in the interview). An analysis from the perspective of structure, components, and details shows that this is exactly what he does.

The statement contains an ‘if and only if’ sentence, and most often each implication is shown separately. In addition, the statement involves an equality sign between two sets,  $S = \text{int}S$ . Proving an equality sign between sets is often (also) done in two steps,  $S \subseteq \text{int}S$ , and  $S \supseteq \text{int}S$ . Proving the right implication ‘ $\Rightarrow$ ’ thus includes two things, whereas the left implication only implies one:

- $S \text{ open} \Rightarrow \begin{array}{l} S \subseteq \text{int}S \\ S \supseteq \text{int}S \end{array}$
- $S \text{ open} \Leftarrow S = \text{int}S$ .

The proof of the statement (in the case where  $S$  is not the empty set<sup>5</sup>) is provided below. Since the textbook used in the course only states the theorem and not the proof, the following proof is constructed by me as inspired by the professor’s review of it. The statement  $S \supseteq \text{int}S$  is always true independently of the set being open or closed.<sup>6</sup> The first three steps contain the proof of the first sub-statement ( $S \text{ open} \Rightarrow S \subseteq \text{int}S$ ). The second sub-statement ( $S \supseteq \text{int}S$ ) is evoked in step four, whereas the third sub-statement ( $S \text{ open} \Leftarrow S = \text{int}S$ ) is proved in the three remaining steps (step 5 to 7):

#### Proof

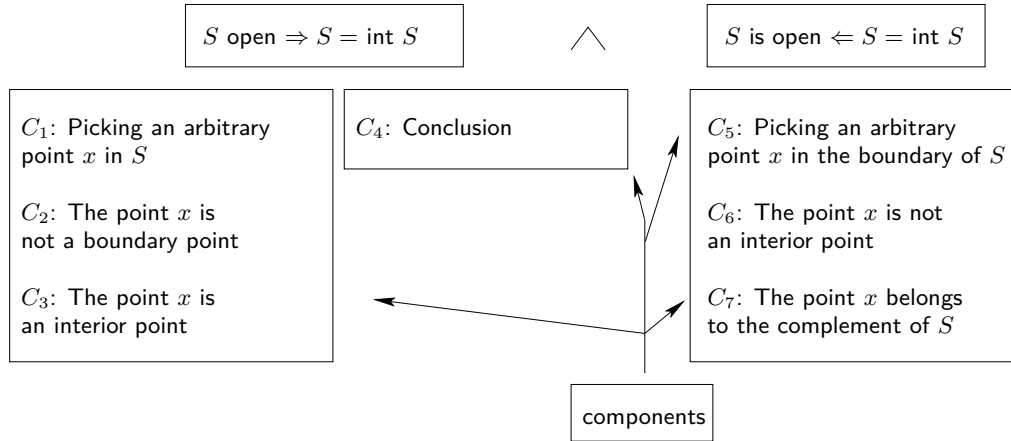
[1] Assume  $S$  is open. Let  $x \in S$ . [2] Since  $S$  is open,  $\partial S \subseteq S^c = \mathbb{R} \setminus S$ , so  $x$  is not a boundary point. [3] Hence, there exists a neighbourhood  $N_\epsilon(x)$  of  $x$  such that either  $N_\epsilon \cap S = \emptyset$  or  $N_\epsilon \cap \mathbb{R} \setminus S = \emptyset$ . Since  $x \in N_\epsilon \cap S$ , this means that  $N_\epsilon \cap \mathbb{R} \setminus S = \emptyset$ , so  $N_\epsilon \subseteq S$ . By definition  $x$  is an interior point of  $S$ . This proves that  $S \subseteq \text{int}S$ . [4] Since  $\text{int}S \subseteq S$ , we conclude that  $S = \text{int}S$ . [5] Conversely, assume that  $S = \text{int}S$ . Let  $x$  be a boundary point of  $S$ . [6] Since  $x$  is a boundary point,  $N_\epsilon \cap \mathbb{R} \setminus S \neq \emptyset$  for any neighbourhood  $N_\epsilon(x)$  of  $x$ . Hence, there does not exist a neighbourhood of  $x$  which is contained in  $S$ , so  $x$  is not an interior point of  $S$ . [7] It follows from the assumption that  $x \notin S$ . This means that  $x$  belongs

<sup>5</sup> If  $S$  is the empty set the statement is trivially fulfilled.

<sup>6</sup> The proof: let  $x$  be an interior point of  $S$ . Then there exists a neighbourhood  $N_\epsilon(x)$  such that  $N_\epsilon \subseteq S$ . Since  $x \in N_\epsilon$  this entails that  $x \in S$ , so  $\text{int}S \subseteq S$ .

to the complement  $S^c = \mathbb{R} \setminus S$ , which proves that  $\partial S \subseteq \mathbb{R} \setminus S$ . By definition,  $S$  is open.  $\square$

The proof structure related to the chosen proof strategy is shown in figure 5.8. Although the theorem appears almost self-evident, the proof structure clearly is not.



**Figure 5.8** The structure of the proof of the statement  $S \text{ open} \Leftrightarrow S = \text{int} S$  is composed of the main steps or components that the chosen proof strategy leads to. Proving the right implication entails justifications that  $S \text{ open} \Rightarrow S \subseteq \text{int} S$ , which is done in components  $C_1$  to  $C_3$ .  $C_4$  concludes by evoking that  $S \supseteq \text{int} S$ . Components  $C_5$  to  $C_7$  justify the left implication.

The chosen proof strategy (which involves a sub-strategy for proving that two sets are equal) combined with the respective definitions explains the different components. To show that one set is a subset of another is done by picking an arbitrary point in the first set and showing that the point belongs to the other set. So the first component is explained or realised through the chosen proof strategy together with the sub-strategy. To realise the second component, that  $x$  is not a boundary point, demands the activation of the definition of an open set. The realisation of the third component demands the formulation of what it means *not* to be a boundary point as well as the activation of the definition of an interior point together with the observation that the intersection of  $N_\epsilon(x)$  and  $S$  is non-empty. If the statement  $\text{int} S \subseteq S$  is justified beforehand, the realisation of the fourth component only demands the recollection of this result. The fifth component is very similar to the first component, but in order to realise it the definition of an open set (and the overall proof strategy) has to be consulted. The sixth component demands the activation and comparison of the definitions of a boundary point and of an interior point. To realise the seventh and final component, the sixth component and the assumption have to be combined.

The professor emphasises the overall proof structure by asking what is meant by ‘iff’. He does not clarify how to show that two sets are equal, but assumes

that the students know this already. He starts by addressing component  $C_4$  (lines 1-3), but presumably because of its triviality he does not mention that this result actually needs to be justified. He chooses to prove the right implication first, and assumes that  $S$  is open. He then asks for the definition of openness, and a student provides him with the answer, which is a detail of the second component in the proof structure. The structure could have been constructed differently, such that the definition of openness is placed in the first component (then the first component might have been named ‘consequence of assuming openness’). When constructing the analysis, I found that placing it first would not naturally lead to the component where an element in  $S$  is chosen.

Next, the professor mentions the definition of an interior point, which is used to realise the third component, and hence is a detail hereof. Just before moving on to choose a point,  $x$  in  $S$ , the professor explains this step by referring to the sub-strategy (*“I need to show that if I take something from  $S$ , that that belongs to the interior”*, line 16), so he now provides the details of the first component. He does not explicate his proof strategy: showing that  $x$  is not a boundary point and using that to show that  $x$  is an interior point.

He carefully explains why  $x$  does not belong to the boundary of  $S$ , the details of the second component, before he moves on to introduce a neighbourhood of  $x$ , which is a detail in component  $C_3$ . He concludes – by referring to the definition of an interior point – that  $x$  is an interior point, thus providing a thorough account of the details of this component. Without explicitly referring to the fact that  $\text{int}S \subseteq S$ , he concludes that  $S = \text{int}S$ , hence leaving out the details,  $D_4$ .

In the last part of the proof, the proof of the left implication ‘ $\Leftarrow$ ’, the professor hastily mentions  $C_5$ , and does not provide any explanations for it before he moves on to  $C_6$  without providing the details of this component either. The details of the last component are provided, leading to the conclusion.

#### 5.10.4 Summary

The course in the supplementary study is very professor guided and controlled. The professor’s goal is to show the students that they – at some point in the future – can learn to construct proofs themselves. He believes that by making them see that they can do the incremental steps in more complicated proofs they will experience that they are able to learn to construct larger proofs by themselves. The time-line representations of the lessons show that the professor involves the students during proof construction or reviews of solutions to tasks, and that he (or the students) often repeats previously introduced definitions during these periods. Not visible in the time-line representations is the fact that student activity during these periods is mostly confined to answering questions from the professor in which he asks for the repetition of a known definition or theorem.

The analysis of a proof validation situation shows that the professor clarifies one aspect of the proof structure, but assumes that the students are able to

identify how the structure is composed of the individual components without explicit guidance. He mentions the components, of course, but not with reference to the structure. The components are in a sense also treated as details. The analysis thus verifies that the professor is mainly concerned with the details of the proof which is in line with what he expressed in the interview.

## 6 Characterisation of solution processes

“So our approach right now is to take the first thing and write it down and the next thing and write it down, . . . and then see if something magically appears.” (Student during solution attempt)

The focus of this chapter is on the solution processes of the four teams observed in the main study. The processes are analysed in two different ways. First, Schoenfeld’s protocol analysis tool results in a macroscopic analysis and provides an overview of the solution processes. Second, an analysis from the perspective of structure, components and details (developed in section 4.3) focuses on the mathematical content of the students’ reasoning and justifications. Due to space limitations, I have chosen to only present the analyses of the solution process of teams A and B. An English translation of almost the entire process of team A is presented in section 6.3.2. The protocol analysis of the solution process of team A also refers to this section. The process of team B is presented more sporadically. The protocol analysis has been included in the margin of the excerpts such that the reader is able to see where in the process the different excerpts are placed. The chapter ends with a summary of the students’ solving difficulties as observed in each of the four solution processes.

Before proceeding to the protocol analysis, the task and a solution to the task are presented as well as argumentation for the choice and formulation of the task.

### 6.1 Argumentation for the choice and formulation of task

Choosing a task to base the examination of students’ solution processes on is not a simple undertaking. On the one hand, the task (or tasks) must not be too easy, allowing the students immediate access to it as an exercise, solvable without any deliberation. On the other hand, the task must not be too difficult in the sense that the students are unable to make any progress. These considerations resulted in the following criteria:

- The students are asked to solve one task only, since this allows them to spend more time trying to solve it. Hopefully, this will also mean progress-



ing further in the solution process than the students in teams A and B in the pilot study.

- The task contains as many fundamental concepts and notions presented in the course as possible. A fulfilment of this criterion justifies that the students only work on one task.
- In order not to be too difficult the task contains a statement and the request to prove it.
- It is possible to make an illustration of the situation in the task and then use this illustration to make progress.

The following task fulfils these criteria.

### Task

A sequence of functions  $\{f_n\}$  is said to be *uniformly bounded* on an interval  $[a, b]$  if and only if there exists a number  $M > 0$  such that:

$$|f_n(x)| \leq M$$

for all  $n$  and for all  $x \in [a, b]$ .

- Show that a uniformly convergent sequence  $\{f_n\}$  of continuous functions on  $[a, b]$  is uniformly bounded.
  - Show that the statement is true only if the interval is closed and bounded.
  - Show that the statement is true only if the sequence is uniformly convergent.

The situation in the task can be illustrated as shown in figure 6.1. The task explicitly asks for a proof of a given statement. The statement involves a definition provided at the beginning of the task. If omitted from the task, the students would have to look the definition up in the textbook, where it is listed in task 7.1.5. Consequently, it is not a result the students are supposed to know. The two supplementary requests in the task are meant as eye-openers to provide assistance in solving the task. The formulation of them is misleading, while the statement is not true *in general* if the interval is not closed or bounded or if the sequence does not converge uniformly, but there exists specific cases where the statement is true even if these conditions are not satisfied. None of the students seem to be confused about the formulation of the two supplementary requests. The task could have been made considerably more difficult if it had been formulated as an inquiry, for example, as: what should apply to a sequence of continuous functions on an interval  $[a, b]$  in order for the sequence to be uniformly

bounded? I found this formulation too difficult, and since none of the four teams perceived the task as easy, I believe I made the right decision.

Several central concepts that the students encountered during the course are involved in the task: continuity, convergence of sequences, uniform convergence, function sequences, boundedness of functions and closed and bounded intervals. Besides involving these concepts the proof of the task revolves around the concept of infinity. The fact that the task contains many of the main notions from the course justifies that the students are given one task only.

The task is characterised as a general proof task (see section 5.7, page 181) since it asks directly for a proof and does not contain any specific expressions. In the pilot study, the analysis of the solution processes revealed that the uncertainty about whether the identity function considered was continuous or not caused a great deal of confusion for the weaker students (teams A and B) and I wondered about how they might have reacted had the task stated that the function was in fact continuous. By formulating the task in the main study as a proof task I hoped to prevent confusion related to uncertainty about the conclusion.

A generalisation of the illustration in figure 6.1 forms the basis of the proof. Since the illustration is founded on the concept image of a uniformly converging sequence of functions as ‘a sequence of functions where the tail of the sequence is contained in an epsilon-strip around the limit function’, the proof also centres around this concept image. The proof of the statement can be constructed differently. The numbers in square brackets in the proof indicate the main steps in the proof. More will be said about this in section 6.3.1.

### Proof

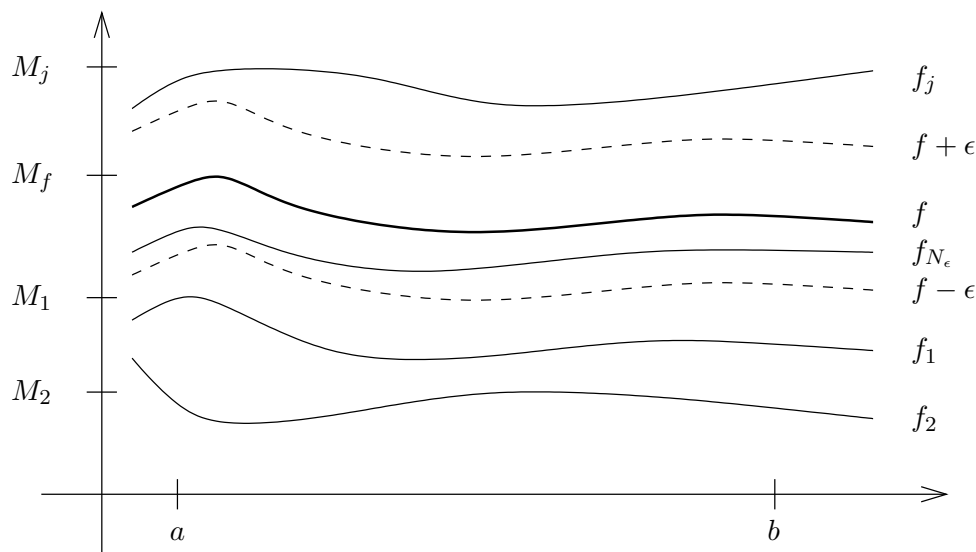
[1] Since the sequence  $\{f_n\}$  converges pointwise, there exists a limit function  $f$  on  $[a, b]$  such that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ . Since  $\{f_n\}$  converges uniformly, given  $\epsilon > 0$ , there exists an  $N_\epsilon \in \mathbb{N}$  such that  $x \in [a, b]$  and  $n \geq N_\epsilon$  imply  $|f_n(x) - f(x)| < \epsilon$ . [2] Since the sequence  $\{f_n\}$  converges uniformly and all the  $f_n$ s are continuous, the limit function,  $f$ , is continuous on a closed and bounded interval, and the Extreme Value Theorem gives that the function obtains its maximum and minimum values in the interval  $[a, b]$ . Thus, there exists a number  $M_f$  such that  $|f(x)| \leq M_f$  for all  $x \in [a, b]$ . For  $n \geq N_\epsilon$  and  $x \in [a, b]$

$$|f_n(x)| = |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)| < \epsilon + M_f.$$

Let  $M_\epsilon = \epsilon + M_f$ . Since each function,  $f_n$ , is continuous on a closed and bounded interval, the Extreme Value Theorem gives that there exists a number  $M_n$  such that  $|f_n(x)| \leq M_n$  for  $x \in [a, b]$  and for each  $n \in \mathbb{N}$ . [3] Choose  $M = \max\{M_\epsilon, M_n : n < N_\epsilon\}$  then:

$$|f_n(x)| \leq M$$

for all  $n \in \mathbb{N}$  and for all  $x \in [a, b]$ . This proves that  $\{f_n\}$  is uniformly bounded. [4] If the interval is not closed and bounded, the functions are not guaranteed to



**Figure 6.1** One possible realisation of the situation described in the task. All functions are assumed to be positive. Given  $\epsilon$  there exists an  $N \in \mathbb{N}$  such that all the functions  $f_n$ ,  $n \geq N_\epsilon$  lie within the epsilon-strip around the limit function  $f$ . The number  $j$  is assumed to be smaller than  $N$ .

be bounded (for instance, the uniformly convergent sequence  $f_n(x) = \frac{1}{x} + \frac{1}{n}$  on the interval  $]0, 1[$ ). [5] If the sequence does not converge uniformly, the maxima of the functions might go to infinity (for instance, the sequence of functions defined on  $[0, 1]$  whose graphs are triangles with bases  $2/n$  and altitudes  $n$ , and zero from  $2/n$  to 1, example taken from [Wade, 2004, p. 185-186]).  $\square$

The proof has been constructed by adding a sufficient amount of rigour to the illustration in figure 6.1. Making an illustration of the task situation would thus bring the students closer to a proof of the statement. I suspected that this feature of the task would provide good opportunities for the students to make progress in a solving situation.

During the task solving observations it became clear that certain aspects of the task formulation had an (non-constructive) influence on the solving processes. It was not specified that the index  $n$  was a natural number. Furthermore, the letter  $M$  was chosen because this letter is normally used to symbolise an upper bound and it was also used in the definition of a uniformly bounded sequence in the textbook. The letter  $M$ , being used in many of the convergence test theorems for series, triggered strategies based on the identification of similarities [Lithner, 2003].

## 6.2 Time-line representations of solution process protocols

The protocol analysis tool developed by Schoenfeld was applied in the analysis of the teaching practice in section 5.3. Instead of the ten categories, I now use the categories proposed by Schoenfeld to characterise solution processes.

### 6.2.1 Schoenfeld's protocol analysis tool

Schoenfeld developed the graphical time-line representation of solution process protocols as a tool in the study of students' metacognitive abilities (thinking about one's own thinking) and their abilities to assess their problem solving processes [Schoenfeld, 1985]. The graphical representation tool only depicts macroscopical aspects of a solution process such as decision making, monitoring and evaluation, while aspects such as student understanding of mathematical concepts involved in the problem or the quality of their reasoning are not visible in the graphical representations.

Each problem solving protocol is divided into episodes or stages characterising student behaviour in one of the following ways [Schoenfeld, 1985, p. 297-301]:

- **Read** This stage includes everything related to reading the problem, also verbalisations of stated conditions or the question.
- **Analyse** This stage contains analytical reflections about the problem. Here the solver tries to make the problem his or her own, to reformulate the problem or the conditions. Characterised as a thinking stage and not a doing stage, this stage is closely connected to the problem.
- **Explore** In contrast to the analysis stage, this stage contains unstructured searching for ideas or relevant information to be used to reach a solution.
- **Plan** This stage includes explicit planning elements.
- **Implement** In this stage the implementation of verbalised or non-verbalised strategies or plans takes place.
- **Verify** This stage comprises all attempts to verify the complete solution or parts of it.

Between two episodes or stages in the process, transition periods might occur. The transitions can be caused by many different things. A transition period might indicate a break in the solution process where students, for instance, make comments on a meta-level. This could, for example, be exchanges concerning the chosen solving strategy, such as:

**Aaron:** In any case, what we do right now is that we look at the first question without even looking at the next, right? It might be the other way around, that there was a hint in the next one, which made it worth looking at.

**Adam:** Yes.

**Aaron:** But that's not the way mathematical task normally are. It is usually the other way around, that you have to use what you have just found out. (Team A, main study)

Schoenfeld's study concerns students who participated in his problem solving course. The tasks the students are being asked to solve are (open) problems (and not proof tasks) and the students do not have any textbooks to refer to or to use during the solving process.

In the solution processes I study, the students are allowed to use the textbook, which plays an important role in the process. Episodes where the students browse through the textbook in the hope of tripping over some kind of information (definitions, theorems or examples) to use to reach a solution are categorised as exploration episodes. Incidents where students browse through the textbook to find pieces of information they recall something about are structured behaviour and might often be part of an analysis or an implementation episode. If shorter periods of time where the students look in the textbook are not contained in stages of exploration, these incidents are denoted by an arrow with a diamond in the time-line representations, while the abbreviation TB (textbook) is used in the excerpts.

A time-line representation of the analysis (corresponding to the time-line representation of the teaching practice in section 5.3.1) gives an overview of the analysis and the solution process. A time-line representation of the analysis of team A's solution process can be found in figure 6.2. The time-line representations of the analysis of all four solution processes can be found in appendix E.

The individual periods are indicated in the margin of the English translation of the solution process protocol of team A provided in section 6.3.2, and the reader is referred to this section to see the specific content of the different periods in more detail.

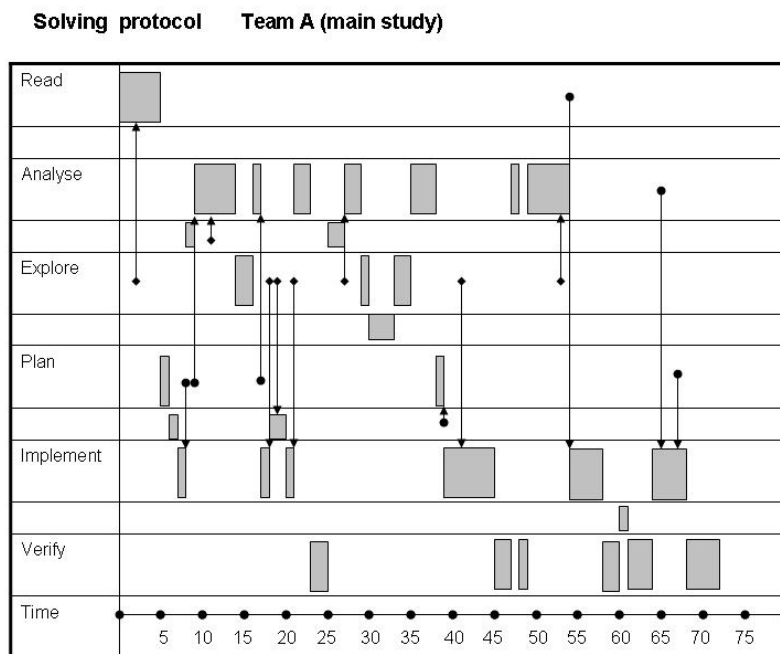
### 6.3 Examining the hypothesis

After having used the protocol analysis tool to provide overviews of the four solution processes, it is now time to take a closer look at the way the students reason. This is done through the framework of structure, components and details presented in section 4.3. All four solution processes have been analysed from this perspective, but due to space limitations I will only present the complete analysis of team A's solution process, and parts of the analysis of team B's solution process. Since the solution processes of teams C and D were highly influenced by a lack of sufficient resources, I have chosen not to present the structure-components-details analysis of their solution processes.

As was the case in section 4.3, the analysis of the solution process takes its point of departure from an analysis of the structure of a proof of the statement.

#### 6.3.1 The proof structure

The definition of uniform boundedness aside, the task is of the type 'if <certain premises> then <conclusion>'. There are three premises and one conclusion:



**Figure 6.2** Time-line representation of solving protocol A. The spacing between the six different categories represents transition periods. Arrows starting with a diamond as opposed to a circle indicate that the students are searching directionlessly in the textbook.

$P_1$  : The sequence  $\{f_n\}$  converges uniformly.

$P_2$  :  $f_n$  is a continuous function for all  $n$ .

$P_3$  :  $f_n$  is defined on the interval  $[a, b]$  for all  $n$ .

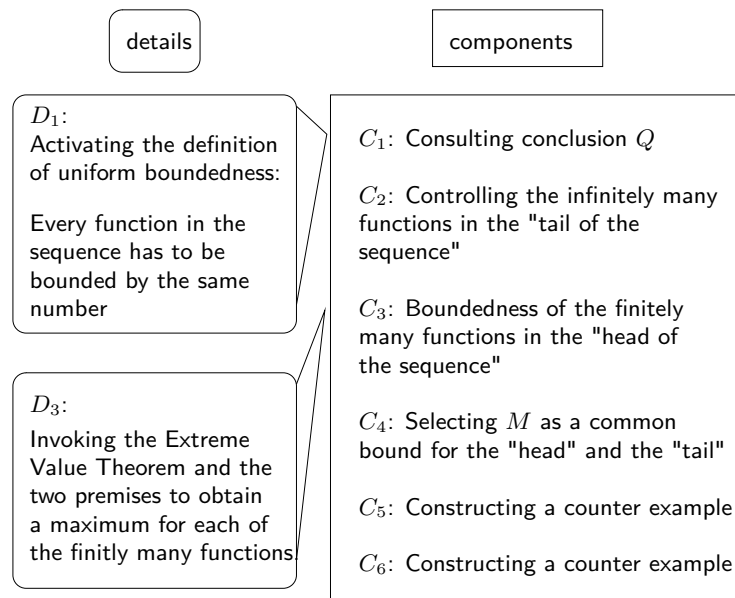
$Q$  :  $\{f_n\}$  is uniformly bounded.

The logical structure of the statement is ‘if  $P_2 \wedge P_3 \wedge P_1$  then  $Q$ ’. The proof of the statement presented in the beginning of this chapter (see section 6.1) does not completely illustrate the process of the proof construction. A hypothetical expert might reason in the following way:

I need to show that all the functions in the sequence are bounded. All the functions in the sequence are bounded, but it might be a problem to select a common value since there are infinitely many functions in the sequence. But because the sequence converges uniformly, the tail of the sequence is controlled in an epsilon-strip around the limit function. Then I only have to choose the maximum value of finitely many functions, and that is not a problem. This means that I just have to select the biggest bound of the

finitely many functions in the head of the sequence, and the bound for the tail. (Hypothetical expert)

The proof construction process starts with the activation of the conclusion. The conclusion,  $Q$ , means that in order for the sequence to be uniformly bounded each of the infinitely many functions must be bounded by the same number,  $M$ . The job is thus to show that all the infinitely many functions are bounded and bounded by the same number. This first step (the first sentence in the imaginary process) is not included in the proof, so the steps in the proof (starting on page 207) correspond to components  $C_2$  to  $C_6$ . The components and thus the structure of the proof are shown in figure 6.3.



**Figure 6.3** The structure of the proof of the claim is composed of the main steps or components that the chosen proof strategy leads to. The realisations of the components are the details of the proof.  $Q$  stands for the conclusion  $Q$ :  $\{f_n\}$  is uniformly bounded. The details of the first and third component are provided as illustration.

This reasoning provides the structure of the proof, but not the details. The realisation of the six components represents the details of the proof production process. The first component requires that the student is able to interpret what is necessary and sufficient to obtain the conclusion. Uniform boundedness demands that every function in the sequence has to be bounded by the same number.

The second and most complex component in the proof entails that the student realises that since the tail of the sequence contains infinitely many functions, it is not possible to show that every function is bounded by the same number just by arguing that each function is bounded. In order to control the tail, a result

about the continuity of the limit function of a uniformly converging sequence of continuous functions (theorem 7.9 in appendix C.2) must be evoked. But this is not all. Each of the three premises have to be applied as well.

The realisation of the third component demands that the solver activates both premise  $P_2$  and  $P_3$  and the Extreme Value Theorem. Even though the realisation of this component also involves several premises and a known result, the step does not include infinitely many functions. Moreover, the professor has several times made use of the theorem (a continuous function on a closed and bounded interval attains its extreme values). As a result, the third component is not perceived to be as difficult as realising the second component. The two times the Extreme Value Theorem is used in the proof (once in component two and once in component three), the student has to realise that the theorem also applies to the absolute value of a function. Either the student can reason that since a continuous function on a closed and bounded interval achieves its extreme values, the maximum of the absolute values of the two extreme values will be the maximum value of the absolute value of the function, or the student may simply argue that the absolute value preserves continuity such that the Extreme Value Theorem can be applied directly to the absolute value of the function.

The realisation of the fourth component demands that the student combines the results obtained so far and formalises the choice of  $M$  as is the case in the proof.

The two remaining components take care of the two supplementary requests. Two appropriate counter examples have to be constructed.

### 6.3.2 Team A

The solution process of team A in the main study clearly shows how the students focus on details and are reluctant to pay explicit attention to the structure as well as the influence their behaviour has on the solution process. The reason that these issues are particularly clear in the process of team A as opposed to the other teams' processes is that the students show they, to a great extent, possess the necessary mathematical resources to construct the proof. The effect of the lack of focus on the structure is easier to detect when the picture is not disturbed by insufficient mathematical resources.

In the following excerpts, notes in the margin indicate when the students directly and indirectly refer to components and details. How they talk about details and components and the consequences their discussions might have on the solution process will be elaborated on in the intermediate sections of analysis. The two students' notes can be found in appendix F. A few parts of the solving protocol have been omitted. Using the line numbers it is possible to see how many lines have been skipped.

- 2      (The two students read the task; their textbooks are closed)  
       **Aaron:** It sounds like mathematics, doesn't it Adam?

Read



TB

- 56 **Adam:** Yes, okay, yes, eh ... but it becomes a little bit more important  
 here, because it's the function value of some sequence  $f_n$  [Mmm] or  
 the function  $f_n$ . This means that there exists an  $M$ , where all the  
 58 functions in the sequence is smaller than.  $C_1$
- Aaron:** Mmm.

First of all, the task is a problem for the two students, and not an exercise. The task appears to be familiar to Adam (lines 47-48), but not familiar enough that he knows what to do. The task is not an exercise for Aaron either. He even shows some confusion about the fact that the sequence consists of functions and not of numbers (lines 16-20 and lines 49-54), so he has not acquired a concept image of a sequence of functions by attending the lectures, but he is very quick to form one (*"sort of a sequence of sequences"*).<sup>1</sup> The mathematical actions that the students apply to construct a proof are, to write down the definition and the conditions and to 'see if they can make something appear' (lines 27-29). Although it is reasonable to restate the definitions of the concepts involved (the first part of their strategy), it is, in this case, not possible to deduce the proof just by comparing the definitions. Hence, according to the definition of a problem/exercise used in this dissertation (see page 4), the task is a problem and not an exercise for the two students.

During the first five and a half minutes (lines 1-59), Adam circles around the connection between uniform boundedness and 'ordinary' boundedness (line 8 and line 48). He starts by connecting the details of the first component, which are the details of the definition of uniform boundedness, with the known definition of a bounded function. He ends up formulating that the definition entails that one number,  $M$ , should exist for which all the functions are smaller than (lines 57-58). He uses his own words to reformulate the definition provided in the task, but he does not conclude that this means that they have to show boundedness of all the functions,  $f_n$ , in the sequence.

- 60 **Adam:** And you could start by writing down, eh, uniform convergence.  $D_2$ , plan  
**Aaron:** Exactly.
- 62 (20 sec. pause. They write down the definition of uniform convergence)
- Aaron:** Well, we don't know what it converges to, but eh.
- 64 **Adam:** Mmm, mmm. ... but that doesn't mean that you can't write it  
 down. It can sometimes be a help to write it down..
- 66 **Aaron:** Yes, of course.
- Adam:** ..write down the two things and then see if.
- 68 **Aaron:** And that is the one which we should be able to recite in our sleeps,  
 right? (He laughs because it is a reference to a comment from the Break  
 professor. He looks in the textbook) I can't remember it, can you?
- 70 **Adam:** This one? (He points at the textbook) I remember that one.

<sup>1</sup> The video-recordings of the teaching verify that Aaron was present the day the notion of a sequence of functions was introduced. In fact he participated in a discussion about the difference between pointwise and uniform convergence.

	72	<b>Aaron:</b> Is it there? (They laugh) Of course you can when it's right there. (Aaron opens the textbook) 'Doesn't know uniform convergence'. (A
	74	comment addressed to STO)
		<b>STO:</b> I am only writing down what happens, not..
	76	<b>Aaron:</b> Yes, of course (ironically) 'You fail'. (They laugh)
Implement		<b>Adam:</b> Okay, let's see.
	78	(40 sec. pause. They write down the definition from the textbook)
		<b>Adam:</b> Okay.
	80	(1 min. pause. They lean back)
$D_2, D_3$		<b>Aaron:</b> So we should use that $f_n$ is a sequence of continuous functions, right?
	82	That is, that they are continuous, right?
		<b>Adam:</b> Yes (hesitant).
Plan	84	<b>Aaron:</b> So our approach right now is to take the first thing and write it down and the next thing and write it down or what?
	86	<b>Adam:</b> Yeah.
		<b>Aaron:</b> And then see if something conjures.
Break	88	<b>Adam:</b> Yeah, I am just wondering if, if you in some way could..
	90	<b>Aaron:</b> In any case, what we do right now is that we look at the first question without even looking at the next, right? It might be the other way around, that there was a hint in the next one, which made it worth looking at.
	92	<b>Adam:</b> Yes.
	94	<b>Aaron:</b> But that's not the way mathematical task normally are. It is usually the other way around, that you have to use what you have just found out.
Plan	96	<b>Adam:</b> Well, I was just considering if there is a way to splice, eh, together the definition of uniformly bounded with uniform, eh, convergence.
	98	<b>Aaron:</b> Well, that's what we have to show.
	100	<b>Adam:</b> Yes.
		<b>Aaron:</b> So you're right there. But that's what I mean. If we..in principle, it should be such that we could write it down, the definition of conver- gence, and then because they are continuous, and then by looking at the definition of a uniformly bounded function, then we could go from the first two step to the next. So you are completely right, we have to splice together..go from one to the other.
	102	
	104	
	106	
Analyse	108	<b>Adam:</b> Yes, exactly. Yes. And we have to start with unifrom convergence and then reach uniform boundedness. ... And then I am wondering, could we use this $M$ for anything?
	110	<b>Aaron:</b> Well, okay. We know uniform convergence. Do we then know any- thing about..and we know something about continuous functions on $D_2, D_3$ closed intervals, they are bounded, right?
	112	<b>Adam:</b> Mmm, yes.
	114	<b>Aaron:</b> And then we know..we know something about, eh, and that's where we sort of have to go, right? But we don't really have..yes, it say here a closed interval from $a$ to $b$ , right? (He points at the task)
	116	<b>Adam:</b> Yes. It was actually interesting, eh, that they are bounded, be- cause..continuous functions, because together with uniform conver- gence, this mean that..then it applies to the whole interval, right?
	118	If we find some epsilon, then it applies to all $x$ in this interval.
	120	<b>Aaron:</b> Yes, exactly.

$C_1$	122	<b>Adam:</b> So what if we..eh..I was thinking about looking up, if there isn't	
		something about bounded that you could see (Adam drops his calcu-	
	124	lator on the floor)	
		<b>Aaron:</b> Now you don't have to stress, we have an hour and a half.	Break
	126	<b>Adam:</b> Well, okay. Good enough.	
		(They look in the textbooks)	TB
	128	<b>Adam:</b> We are almost there, Aaron, I can feel it.	
		<b>Aaron:</b> It's going to be a nice exam. You can solve all the tasks on the first	
	130	day, and then you can take four days off. That's just going to be so	
		nice (meant ironically).	
	132	(30 sec. pause. They look in their textbooks and at the task)	

Without being totally sure about what they are supposed to show, they start discussing a possible solving strategy. Aaron says that “*in principle*” it should be possible to reach a solution by writing down and comparing the definition of uniform boundedness on the one hand with the definition of continuity and uniform convergence on the other hand (lines 101-106). Adam agrees. He has just suggested that they should try “*to splice the definition of uniformly bounded with the property of being uniformly convergent*” (lines 97-98).

Since the solution does not immediately jump out when they compare the different definitions, they proceed by trying to remember results about continuous functions and convergent sequences (lines 110-112). This makes Adam remember that continuous functions on closed and bounded intervals are uniformly continuous (he does not express it this clearly) (lines 117-120). They are thus focusing on the details in the proof, which is obviously connected to the conditions or premises stated in the task. At one point, Adam tries to focus on the conclusion, and suggests that they should try to find results in the textbook related to boundedness (lines 122-123). This might have been a productive strategy, but unfortunately he drops his calculator on the floor and although he starts looking in the textbook, it seems that he is unsure about what he is looking for. Focusing on the boundedness of the functions instead of trying to find out what they can deduce from the conditions is a way of focusing on the structure of the proof instead of on the details.

			Analyse
		<b>Aaron:</b> What we have to show..if we can show that the thing about $f_n$ tends	
134		to $f$ , right, like we have written as the definition. Well, if we can show	
		that $f$ is a continuous function, because all $f_n$ are – and I think we can	
136		– then we have basically..and then we could say something about..and	
		we know that it's on a closed and bounded interval, then we should	
138		be able to..then we know from what I said before, then it's bounded.	$D_2$
		And then it's just a question if it applies to all functions, that's just	$C_2$
140		... do you follow what I am saying?	
		<b>Adam:</b> Yes, yes, eh. And then compare it to the fact that we have uniform	
142		convergence..	
		<b>Aaron:</b> Yes, exactly.	
144		<b>Adam:</b> ..towards this function, right?	

- $C_1$
- 146 **Aaron:** Yes, exactly. So if the fact that  $f_n$  is continuous can be passed on  
to  $f$ , and the interval is closed and bounded, then we should be able  
148 to..then we should be able to show that there exists an  $M$  which is big  
enough, right?.
- 150 **Adam:** Yes, so it's in fact ... what's it called ... bigger than, eh, the whole  
sequence and the limit for all  $x$  in  $[a, b]$ , right?
- Aaron:** Mmm.

Because they have not clarified properly the first component, Aaron mistakenly thinks that they also are asked to show that the limit function is bounded (the definition of uniformly boundedness does not in fact say anything about the limit function). After having correctly reasoned that it is possible to show that  $f$  is bounded, he speculates how they should prove that “*it applies to all the functions*” (line 139). To show that  $f$  is bounded is a means of showing that the sequence is bounded, but it is in fact not the goal. The students are not clear about the difference, and during the process they alternate between believing that they have to show that  $f$  is bounded and realising that this is not what they are asked to do (but it is of course a necessary detail in the proof). When Aaron repeats his strategy (lines 145-150), it might at first seem that he is close to a solution, but he has just added another piece of information, that  $f$  is bounded, and although it is a very relevant piece of information, they cannot use it yet, because they have not realised the purpose it serves (this demands  $C_2$  and not just the details  $D_2$ ). But Aaron manages to formulate their goal (lines 147-148), and Adam's answer confirms that he too has identified it (lines 149-150). So after about fourteen minutes, the first component is – more or less – in place.

The excerpt shows that Aaron formulates a need to find or prove a result stating that the limit function is also continuous. Since he is able to formulate this, they are able to realise the relevance of the result when they encounter it in one of their textbook explorations later in the process.

- Explore 152 **Aaron:** I think we have to browse a bit. (they browse through the textbook)
- 154 **Adam:** Yes. (He is still looking at the task) And there is also .. no, there is  
not. If they are continuous, not, it's damn ...  
(Aaron browses, Adam underlines something in the task, 40 sec. pause)
- 156 **Aaron:** Well, according to this theorem 3.39, right?
- Adam:** Eh.
- 158 **Aaron:** It was just something we talked about before. On page 81, there we  
have that if it's continuous on a closed and bounded interval, then it's  
160 also uniformly continuous, right? (Adam nods) And here we exactly  
have that  $f_n$  are continuous functions on a closed interval, right?
- 162 **Adam:** Then it's simply uniformly convergent?
- Aaron:** Continuous.
- 164 **Adam:** And, yes. Continuous. Eh, ... if we then could find the other one.  
(They look in the textbooks, 15 sec. pause)
- Analyse 166 **Aaron:** But that thing that there is a new function for every  $n$ , right? If  
you look (Aaron points at the task)
- 168 **Adam:** Yes.

170	<b>Aaron:</b> For every $n$ we have a new function, right, and each one is continuous, right? But then it has something to do with the fact that it's uniformly convergent, then it tends to some function $f$ . Because we	
172	can't..we do not know anything about the individual $f_1, f_2, f_3$ and so on. But because it's that uniformly convergent, then it tends to $f$ and	
174	if we can pass this continuity to that, then I suspect – and that was what we talked about before – that we have the same again with a	
176	continuous function on a closed interval..or sequence or something.	
	<b>Adam:</b> Yeah.	Plan
178	<b>Aaron:</b> I think this is the right strategy, Adam.	
	<b>Adam:</b> I think so too.	
180	<b>Aaron:</b> Then we just have to find some mathematics, then we're home free.	
	<b>Adam:</b> Eh. There is something here, no, it doesn't say anything about being closed. That's not fair. Now we're just about to harvest some	Implement,
182	theorems.	TB
184	(Adam browses through the textbook, 15 sec. pause)	Break
	<b>Aaron:</b> Well, what's bothering me the most about this, right, it's not that..it's just that we can't figure it out like that. Because what we're saying	
186	should be easy for us to formulate if it's correct.	
188	<b>Adam:</b> Yes. But at the same time I feel that there are different things that we could start with and begin to calculate, right?	
190	<b>Aaron:</b> Yeah, yeah, yeah. That's why I have..well, I don't know.	
	<b>Adam:</b> But it's..if you first start to calculate, then half an hour is used quickly. Where we don't talk so much. Eh, I'll try..	
192	<b>Aaron:</b> The question is if we're at a point where Michael (the professor) thinks that you have to know the whole textbook and be able to calcu-	
194	late, then you aren't challenged. Then it's not difficult at all. Weren't you there, when he said that?	
196	<b>Adam:</b> What? That it's not enough to know the textbook?	
198	<b>Aaron:</b> At the exam.. at the exam you have to..well, you just have to know it all, it's not that difficult. And he might be right there.	
200	<b>Adam:</b> It's certainly not all wrong. (They browse through the textbook)	
	<b>STO:</b> If you feel a need to calculate separately and not talk, then its' also..well, it's not.	
202	<b>Adam:</b> Yes. We just have to show in some way, that..because the only thing that we're told is that it's a uniformly convergent sequence, so in some	
204	way this has to lead to (They laugh) uniformly boundedness. On a bounded interval. There has to be something that says something,	
206	because it's obvious, if it's unbounded then it can't be uniformly continuous. Uniformly convergent, it's called. All these concepts, I can't	
208	figure them out.	
210	(Aaron writes, 20 sec. pause)	Implement
	<b>Aaron:</b> I am just writing what the definition is.	
212	<b>Adam:</b> Yes.	
	(Aaron writes, Adam looks in the textbook, 1 min. 25 sec. pause)	

During this period of exploration, Aaron trips over a theorem stating that a continuous function on a closed and bounded interval is also uniform continuous. This information is not actually needed to construct the proof and might very well create confusion later on in their process, since Aaron keeps referring to the fact that they have full control over the  $xs$ .

Aaron summarises their analysis of the situation and sketches a possible proof strategy (lines 169-176). This strategy clarifies that they need a result claiming the limit function is also continuous, because then they can use the Extreme Value Theorem on this function as well. Aaron and Adam's later comments show that they have the idea that a solution demands "*some mathematics*", and that it is obtained partly by "*harvesting*" some theorems and partly by making "*calculations*" (line 180, line 182, and line 189).

Although Adam has a feeling that there are several directions they could follow, which I interpret to mean Adam has a sense of the structure (lines 188-189), it seems nevertheless that they are trying to construct the proof by considering all the components at once. Not being able to cut up the proof in smaller parts – which is something that demands an understanding of the structure – makes it difficult to provide the details (= "*some mathematics*").

Analyse	214	<b>Adam:</b> It really looks a lot like ... the definition of a normal, not a sequence, of functions, just a bounded function.
	216	<b>Aaron:</b> Yes, yes, it's exactly the same.
		<b>Adam:</b> The only difference is really just that it's a sequence.
	218	(10 sec. pause)
$D_2$		<b>Adam:</b> Oops, if $I$ is a closed and bounded interval and $f$ is continuous on $I$
	220	- and it's ... then $f$ is bounded on $I$ . That's a start. (Aaron looks at the page) Or what?
	222	<b>Aaron:</b> Yes, exactly, that was what we needed.
		<b>Adam:</b> But eh.
	224	<b>Aaron:</b> I just have to, eh.
$D_3 \rightarrow C_3$		<b>Adam:</b> This means that because all these functions in the sequence are
	226	continuous, they are also bounded because the interval is..
		<b>Aaron:</b> Yes, yes, but that was what we said before.
	228	<b>Adam:</b> Now we're getting somewhere.
		<b>Aaron:</b> Yes, exactly, that was what we started saying. What we just have
	230	to show is that this continuity is inherited when it converges, because
		it's not $f_n$ we're interested in, it's what it converges to, right?
	232	<b>Adam:</b> It says here 'show that a uniformly convergent sequence', it doesn't
		say anything about the limit value..
	234	<b>Aaron:</b> No, that's right, why are we then sitting..this is what I said from
		the beginning, so you should object.
	236	<b>Adam:</b> Because..
		<b>Aaron:</b> Yes, it's true. It's actually the other way we have to show, that's
	238	irritating.
$C_3$		<b>Adam:</b> Because, if we then say, well, then all the functions in the sequence
	240	must be bounded, right, and then we look at..
Verify		<b>Aaron:</b> But then we have practically said it all, right?
$C_3$	242	<b>Adam:</b> Yes. Because then it's just a question of looking at definition 3.25,
		which says that $f$ is bounded on $E$ , then it means this, right [Yes, yes]
	244	Then it's just $f_n$ is smaller than $M$ and then we're practically there,
		aren't we?
	246	<b>Aaron:</b> Yes, now, now it sounds right.
		<b>Adam:</b> I think so.
$C_4$	248	<b>Aaron:</b> Then you just have to choose the biggest $M$ .

(15 sec. pause)

250 **Adam:** Do we agree that it's done now? Can we write it down?

**Aaron:** While you write it down, I just have to understand this completely.

252 **Adam:** Okay.

**Aaron:** Because it seems to simple.

254 **Adam:** Yes. I feel that too. But there are two more questions. It's just  
because we are so clever, Aaron.

256 **Adam:** Yes, okay, chill out, Adam. But then you don't really use that it's  
uniformly bounded, do you?

258 **Adam:** Shh.

**Aaron:** Now you're just saying that a sequence of continuous functions is  
uniformly bounded.

260 **Adam:** Okay, you may have a point.

The next period ends with the identification of the third and fourth components (lines 225-228 and line 248), such that the proof now consists of  $C_1$ ,  $C_3$ , and  $C_4$ . Aaron has doubts about the result, because “*it seems too simple*” (line 253). He is not basing his assessment on mathematical considerations, but on the fact that they have not used all the premises given. In order to remove this obstacle, Adam suggests that they do not have to use the particular premise until the second supplementary request.

The preceding break lasting three minutes has been omitted, twenty-seven minutes have passed.

396 **Adam:** Do you then have any idea about why this is only true if the sequence  
is uniformly convergent?

398 **Aaron:** Nope, not when the other isn't true. But I was wondering ... we  
have, we have, what's it called, it's a bit strange, but we have some  $x$ s  
as well as some  $f(x)$ s.

400 **Adam:** Yes, we have to keep track of two things, right?

402 **Aaron:** And you could say the  $x$ s should be controlled by this uniform con-  
vergence on a closed interval [Exactly] and then there is this..what  
happens when  $n$  tends to infinity? Nothing much, because it's uni-  $C_2$   
formly convergent. It's only this we need.

404 **Adam:** The advantage of uniform convergence is exactly that if we find some  $D_2$   
epsilon such that  $f_n - f$  is smaller than epsilon, well we have control  
over all the  $x$ s in the interval, right? [Mmm] Whereas if it was only  
pointwise, then we only knew about one individual  $x$  ... and of course  
if we didn't have uniform convergence, then we could not control the  
 $x$ s any more.

410 **Aaron:** No.

412 **Adam:** But I don't think that we can write that. There has to be uniform  
convergence so we can control the  $x$ s, ergo.

414 **Aaron:** Yes, but that's it.

**Adam:** Then we just have to squeeze it into some mathematics. That's just  
nice work.

Since they have to prove the supplementary requests anyway, they start to wonder about why the sequence has to converge uniformly (line 395). Aaron



explains that they have to operate with  $x$  in the interval  $[a, b]$  as well as the functions  $f_n(x)$ , and that the condition of uniform convergence of the sequence has something to do with controlling  $x$  for the individual functions in the sequence. What he might mean is that uniform convergence controls the function values of each  $x$  in the interval. Aaron repeats that they need uniform convergence to control the  $x$ s (line 408) and not the function values, but I still find it most likely that they mean that they need to control the function values, and that they are on the right track regarding the purpose of demanding uniform convergence. Aaron actually manages to identify the second component, “*what happens when  $n$  tends to infinity?*”, and he is also able to conclude (although without providing proper justification) that it is not a problem since the sequence converges uniformly (lines 402-404). Adam reveals that it is difficult to write down their hunch about what is going on (line 412), but that ‘mathematics’ is needed to provide proper argumentation (line 415).

- |         |     |   |
|---------|-----|---|
| Explore |     | <b>Adam:</b> It’s very interesting what this remark 3.27 says. It says that this  |
|         | 418 | theorem that it’s bounded if it’s continuous, it says that “ <i>the extreme value theorem if either closed or bounded is redrawn from the</i> ” [Mmm] |
|         | 420 | So here we might have something for this one. (He points at the first supplementary request)  |
|         | 422 | <b>Aaron:</b> That might be true, that could be nice.   |
| Break   |     | <b>Adam:</b> I think that we have like a fair sketch for each one, a good sketch  |
|         | 424 | for all of them.  |
|         |     | <b>Aaron:</b> So let’s finish.  |
|         | 426 | <b>Adam:</b> Can we solve them? Get it overwith.  |

They feel close to a solution (lines 423-426). They basically have all the details, but they are missing the structure of the proof, which would provide the justification and thus the meaning of the details. Therefore, they have trouble completing the proof. The rest of the break and the preceding exploration period have been omitted, a time period of seven minutes. Thirty-six minutes have passed.

- |         |     |  |
|---------|-----|--|
| Analyse | 438 | <b>Adam:</b> I am just sitting here looking at a small theorem, which might be related to the last question. (Aaron stops writing and looks in Adam’s  |
|         | 440 | textbook) This one.  |
|         |     | (Aaron looks at the theorem in the textbook)   |
|         | 442 | <b>STO:</b> What number does it have?  |
|         |     | <b>Adam:</b> 7.9. Because, you could say that if..   |
|         | 444 | <b>Aaron:</b> Well, I think you might be right, that we shall use that or could use it.  |
|         | 446 | <b>Adam:</b> Because then it practically means that both..then we know that all functions in the sequence plus the limit..                             |
|         | 448 | <b>Aaron:</b> But that’s basically what I want to show, right there.   |
|         |     | <b>Adam:</b> Okay.   |
|         | 450 | <b>Aaron:</b> Then I will stop trying to show this. (They laugh) Although I was almost done. (He is ironic, they laugh) Well, that’s..aren’t we almost |
|         | 452 | done then? Now we know that, eh, now we know that..  |

454	<b>Adam:</b> When they are continuous, then they are also bounded and we know that the sequence..	$D_3$
456	<b>Aaron:</b> We know that when $n$ tends to infinity..then we also get a continuous sequence on a closed and bounded interval, which therefore is bounded.	$D_2$
458	<b>Adam:</b> Yes.	
	<b>Aaron:</b> Then we have to..if it's..yes, that's it.	
460	<b>Adam:</b> Yes, and then we have this remark to question two, right? This thing, that it's only true if it's closed. The interval, right?	
462	<b>Aaron:</b> Yes, yes. There was some theorem, where we should show that and find an examples where it wasn't true, right? Or what?	
464	<b>Adam:</b> It says that it's definitely false if one of those things are not satisfied.	
	<b>Aaron:</b> Yes, yes. That's what I mean, but we have to find an example where it's not bounded.	Plan
466	<b>Adam:</b> Well, but I think that that must enter into the proof that it's false.	
468	<b>Aaron:</b> Shouldn't we formalise the first answer before we..	
	<b>Adam:</b> Yes.	
470	<b>Aaron:</b> We might have overlooked something.	
	<b>Adam:</b> Yes.	
472	<b>Aaron:</b> So what was the first thing that we needed? Well, that's that continuous functions on a closed and bounded interval are bounded, right? This was the one you took as a starting point, right?	$D_3$
474	<b>Adam:</b> Yes, on page 74. (They open the textbooks)	
476	<b>Adam:</b> You always need like two to three bookmarks when you are solving tasks.	Break
478	<b>Aaron:</b> Yes, you need to be able to use the textbook.	
	<b>Adam:</b> That's right, yes.	
480	(Aaron makes a note, Adam also writes something down, but erases it again, 1 min. pause)	
482	<b>Aaron:</b> Because according to this theorem 3.26 we know that ... what do we know? That's for each $n$ .	Implement $D_3$
484	<b>Adam:</b> Yes, exactly.	
	<b>Aaron:</b> That it's for..how do you write that?	
	<b>Adam:</b> Eh.	
486	<b>Aaron:</b> It's just that $f_n$ is bounded? That's not good enough, then it should apply for all $n$ , and that's what we should prove afterwards, when we have proved that it's true for a fixed value of $n$ .	
488	<b>Adam:</b> Wait a minute. I think I've got it..I just have to.	
490	(Adam writes something down, Aaron looks in the textbook and moves on to browse in the textbook, 3 min. 15 sec. pause)	
492	<b>Aaron:</b> How you got something? How do you write it down?	
494	<b>Adam:</b> Now I have tried to write it down, but it's, I am of course not sure that it's the correct formulation.	
	<b>Aaron:</b> Try to explain it.	
496	<b>Adam:</b> I wrote that according to theorem 3.26 and definition 3.25 there exists an $n$ for every $f_n$ or $f_N$ on an interval on the natural numbers where $f_N$ absolute value, is smaller than $M$ on $\mathbb{R}$ .	$D_3$
498	<b>Aaron:</b> Have you then?	
500	<b>Adam:</b> Now I have written it down, right?	
	<b>Adam:</b> Yes. Okay, that's where you are now.	
502	<b>Adam:</b> Yes.	
	<b>Aaron:</b> (He reads from the textbook) Where $f_N$ , yes.	

- 504 **Adam:** And then I wrote that we have an  $M_n$  for every  $n$  belonging to the  
 $D_3$  natural numbers. That just means that we have a maximum for all ...  
506 the functions, right? And then maximum of this must be smaller than  
 $f_N$  on  $[a, b]$ .
- Verify 508 **Aaron:** Yes, but that's of course some of the steps, but haven't you skipped  
some?  
510 **Adam:** Yes. I most likely have.  
**Aaron:** Because if you show it for all  $n$ , that's what I am wondering about..  
512 **Adam:** I for instance miss involving uniform convergence. (They laugh)  
**Aaron:** Okay. Yes, that's exactly it, it's something about — it has to apply  
514 for all those, for all  $n$ , right? So this theorem 3.26, it's that (he reads  
from the paper)  $I$  is a closed and bounded interval and  $f$  is a function  
516 from  $I$  to  $\mathbb{R}$  which is continuous on  $I$ , then the function is bounded,  
right? ... it then applies to ... to a fixed  $n$ , that is, if it's called  $f_n$ .  
518 ... we don't have one function  $f$  or  $g$ , we have a sequence of functions,  
right?.
- 520 **Adam:** Yeah.  
**Aaron:** And for every of those elements, for every  $n$  you have a function.  
522 **Adam:** Yes.  
**Aaron:** And for each one of them, it's continuous and all of them on a  
 $D_3 \rightarrow C_3$  524 closed and bounded interval. This means, for every..that's what you  
said..there exists an  $M$ , this is what I think.  
526 **Adam:** Yes. And then I thought..  
**Aaron:** But, but, it's just, how do you write it down? Because if it was  
528 correct what we said before [Mmm] then we..then this should be a  
fixed  $n$ , a fixed but arbitrary  $n$  in fact, right? But afterwards we still  
want to show that when  $n$  tends to infinity then it was okay, because  
that was what we should use uniform convergence for.
- $C_2$  530 **Adam:** Yes. And I also thought about if I had to include the limit here and  
Analyse 532 say, well the limit is also because this uniform convergence, well then  
 $D_2$  534 the limit is also a continuous function and include it all in this.  
**Aaron:** Yes. But this is also, isn't it? What I couldn't figure out if it's  
536 redundant to say first for an arbitrary  $n$  and then afterwards let  $n$   
tend to infinity. But that's how you are supposed to do? When you  
538 have a fixed..it's more like something fundamental, right? It's not..how  
you do it.
- 540 **Adam:** It's something about separating..maybe that's where I don't follow.  
To separate the limit from the sequence, if you understand what I  
542 mean?  
**Aaron:** Yes, exactly.  
544 **Adam:** Because it's two separate things, right?  
**Aaron:** Yes, right.  
546 **Adam:** And then I thought, when it says a uniformly convergent sequence of  
continuous functions, then it means that no matter what  $n$  we take, as-  
548 tronomic, no matter how astronomic the number, it's still continuous.  
**Aaron:** Yes, yes.
- $D_2$  550 **Adam:** But that does not guarantee that the limit isn't. If it was just  
convergent.  
552 **Aaron:** Yes, that's the problem.  
**Adam:** And I, even..  
554 **Aaron:** But that's what we have said, we are just not good at formulating  
this in a rigorous way, right? If it's correct.

During a period with breaks and exploration (it is only Adam who explores, since Aaron is trying to show that a continuous function on a closed and bounded interval is bounded), they manage in collaboration to formulate almost the entire proof (lines 453-458). They miss the detail about whether they can choose a common number,  $M$ , for all the bounded functions. After having discussed a solution strategy for the supplementary requests, they decide to formalise what they know about the first request (lines 467-468). In the following periods of implementation and verification, the students focus on the details of the third component. Aaron states that they have to use the condition of uniform convergence to secure the boundedness when  $n$  approaches infinity (lines 527-531). Adam responds by focusing on the details of the second component, and they debate whether it is enough to show that  $f_n$  is bounded for an arbitrary  $n$  or if they also have to consider the case when  $n$  goes to infinity. He argues that maybe they have to look at the sequence separately from the limit, so instead of dividing the sequence into a head and a tail, he ends up focusing on the sequence as opposed to the limit function. Thus, even though Adam, at an early stage of the process (line 232), makes it clear that they are not really interested in the limit function, the only result connecting uniform convergence to the rest of the conditions is that the limit function is continuous. This means that their detailed focus when trying to combine details to reach a conclusion prevents them from identifying that the infinite property of the sequence is what could be characterised as the main point or main problem in the proof.

556	<b>Adam:</b> And I think that, even though we have solved it, I still don't have anything that explicitly explains that because it's uniform convergence,	Verify
558	then we control the $x$ s, If you understand?	
	<b>Aaron:</b> I understand what you mean. Don't you think so?	
560	<b>Adam:</b> I don't know if it's contained in that theorem about the limit is continuous. It might be.	
562	<b>Aaron:</b> Well yes.	
	<b>Adam:</b> Because if it's not, if it wasn't uniformly convergent then we couldn't know if the limit was continuous and then we couldn't know if it had a maximum.	Analyse $C_2/D_2$
564		
566	<b>Aaron:</b> But that's in this theorem 3.26, right? Or what?	
	<b>Adam:</b> Yes, because yes. It's that..	
568	<b>Aaron:</b> Because there it applies for all, for all $x$ s belonging to the closed interval, when it's a continuous function.	$D_3$
570	<b>Adam:</b> Yes.	
	<b>Aaron:</b> Then the function is bounded, right?	
572	<b>Adam:</b> Yes, yes, that's right.	
	<b>Aaron:</b> In this way, you have practically squeezed all the function values for all the $x$ s for a certain one of these functions in the sequence, right?	$D_3$
574		
576	<b>Adam:</b> Mmm. Yes. Yes I also wondered about, if we said that it wasn't uniform conv..if we said that it was just convergent or something, if it was still continuous, then it would mean that the limit function wasn't continuous, then it could happen that all the functions in the sequence, eventhough it's bounded and continuous, then they tend to something	
578		

- 580                   unbounded. This means that the maximum for each function keeps on  
growing, we can never find a finite one [Yes] It's like that.
- 582     **Aaron:** Yes.
- Adam:** Okay.
- 584     **Aaron:** I think so. This could be a suggestion.
- Adam:** So if we also include the limit, then we have..because you can say  
586                   that all the sequences have to point at the limit. If we take all the  
maxima for all the functions in the sequence and the limit, then we  
588                   must have caught it all or what?
- Aaron:** I can't figure it out.

An exchange follows in which the students again show confusion about and difficulties in separating between the functions  $f_n$  and the  $x$ s in the interval. They keep thinking that the condition of uniform convergence affects and controls the function values of the individual functions in the sequence (lines 556-574). Despite this confusion, Adam is able to clearly explain the consequences of not having uniform convergence (lines 575-581), and he summarises their solution: *"If we take all the maxima for all the functions in the sequence and the limit, then we must have caught it all."* (line 588). Again, he separates between the sequence and the limit, instead of the head and the tail of the sequence. Two short periods of exploration and reading lasting four minutes in total have been omitted.

- Implement 604     **Adam:** But I have tried to make a sequence..
- Aaron:** That's a good idea. A specific one?
- 606     **Adam:** A really specific one. Which is defined there. (He points in the  
 $D_4$                    textbook) Which is called  $M_n$  and which is equal to supremum of  
608                    $f_n(x)$  on  $[a, b]$ .
- Aaron:** Why?
- 610     **Adam:** And then I say that supremum of it must be equal to  $f_n$ .
- Aaron:** Yes, yes.
- 612     **Adam:** Or greater than, if you know what I mean, right?
- Aaron:** I can see what you have written and it looks right.
- 614     **Adam:** Good, yes, exactly. Even though it's a bit messy. But this could  
in principle be infinite. But then it's..then I say, because we have  
616                   uniform convergence we know that  $f$  converges and bla. bla. bla. Eh,  
the sequence  $f_n$  is bounded and tends to capital  $M$  equal to supremum  
618                   of  $f(x)$ .
- Aaron:** Eh, okay?
- 620     **Adam:** Because it's continuous because of uniform continuity.
- Aaron:** No, this one, you mix  $f$ ,  $f$ , this one is  $f$  and the other is  $f_n$ , right?  
622                   This one is continuous because of the theorem you had before.
- Adam:** Yes, exactly.
- 624     **Aaron:** Okay.
- Adam:** And when this is continuous, then we know from this theorem that  
626                   this  $M$  and this is this  $M$  becomes..
- Aaron:** Yes, yes.
- 628     **Adam:** And this is what this sequence has to converge to.
- Aaron:** Yes, yes, that's right. You are completely right. But I don't know  
630                   if it's necessary to do it like this, but it might be right.

632 **Adam:** And that's why we know that supremum of  $M_n$  is bounded. Isn't  
it?

634 **Aaron:** But that's something you say from the beginning, this is not the  
one you're supposed to examine, you know sup..you know this,  $M_n$  for  
636 a fixed  $n$ , when you have an  $n$  fixed, then you know that the function  $C_3, D_3$   
is bounded from this theorem 3.26. When  $n$  tends to infinity then  
638 you know that, if we call what it tends to for  $f$ , that it's bounded be-  $C_2, D_2$   
cause, eh, it's uniformly convergent. [Mmm] And therefore, then..and  
640 it's..that's what we showed using this theorem, that it's also conti-  
nuous, and then it must be ... bounded.

**Adam:** Tsk.

642 **Aaron:** What was it that you pulled out before, where you said that if it  
was just in a..because I am doubting if I have read it correctly?

644 **Adam:** But you just have to remember that, eh, that one of those..that  
the limit could be bigger than the supremum of the elements in the  
646 sequence.

**Aaron:** Yes, yes.

648 **Adam:** So you have to take the biggest of the two, right?  $C_4'$

**Aaron:** Or the many..

650 **Adam:** Then we must have caught all the damned functions there are. Verify

**Aaron:** Couldn't you write all the functions down (they laugh), also the  
652 negative  $n$ s and the  $\pi$  half?

**Adam:** Okay. (They are joking around)

654 **Adam:** And then, yes ... and we know that if the interval is not closed then  
our whole argumentation falls a part, because we can't use 3.26.

In order to formalise their reasoning, Adam constructs a sequence of the maxima,  $\{M_n\}$ , of the functions,  $f_n$ . He postulates that the sequence converges toward the maximum of the limit function, and since the sequence converges, it has a finite supremum. This means they can choose the biggest of the two numbers, either the supremum of the number sequence or the maximum of the limit function (line 648). This provides an alternative way to construct the proof than the proof presented. In order to differentiate, I use apostrophes on C and D to indicate components and details in Adam's version. The rest of the verification and a break have been omitted, corresponding to four minutes. One hour has passed.

672 **Aaron:** Well right now, we argue that for an arbitrary  $n$  [Mmm]  $f_n$  is a Verify  
continuous function on a closed and bounded interval. This means  
that for a fixed  $n$ ,  $f_n$  is bounded on  $[a, b]$ .

674 **Adam:** Yes.

**Aaron:** When  $n$  tends to infinity we know that  $f_n$  tends to  $f$ ..

676 **Adam:** Uniformly.

**Aaron:** Yes, uniformly and eh converges uniformly to  $f$  and then according  
678 to this theorem 7.9 it means that, since  $f_n$  is continuous then  $f$  is  
also continuous because it applies for all  $x$ . [Yes] And this means that  
680  $f$  which is a limit function, eh, what's it called, when it converges  
there, it's continuous on all  $x$  on a closed interval. Then we can apply  
682 theorem 3.26, which says that when we have a continuous function on  
a closed and bounded interval, then the function is bounded.

	684	<b>Adam:</b> Yes.
	686	<b>Aaron:</b> Then I guess we have showed for all..for a fixed but arbitrary $n$ tending to infinity, $f_n$ is bounded and that's what we should prove.
		<b>Adam:</b> Mmm.
	688	<b>Aaron:</b> Aren't we done then?
	690	<b>Adam:</b> Yes, I would think so. I just made the sequence and said that the sequence is bounded, because the sequence $M_n$ is bounded, because it converges to a fixed $M$ , which is supremum of $f$ .
	692	<b>Aaron:</b> You have to say that one more time so that my brain can hear it.
$C_3'$		<b>Adam:</b> Well I make this sequence which is defined to be supremum of $f_n(x)$ on $[a, b]$ [Mmm]. But I don't know..well besides if we said that it was not uniform, I don't know if it converges, if this one then is equal to infinity, because $M$ gets bigger and bigger. But when it's uniform I know it. Then I know that the limit $f$ is also continuous, which means that $f$ has an $M$ which $M_n$ necessarily must point to.
$C_2'$	696	
	698	
		<b>Aaron:</b> Yes.
$C_4'$	700	<b>Adam:</b> So that's why I know that $\sup M_n$ is a fixed number and then I can say that it must be..it's the biggest of all the $M_n$ s. It must be..satisfy this.
	702	
Implement		<b>Aaron:</b> Well should we pretend that it's correct and then say, what's next? If the interval is closed and bounded.
	704	
		<b>Adam:</b> Yes, I use that..
	706	<b>Aaron:</b> Okay, are you already done?
		<b>Adam:</b> No, I don't know if I am right, anyway I have said that according to this remark 3.27, it says that if either the interval is not closed or bounded, then this theorem 3.26 is wrong.
	708	
	710	<b>Aaron:</b> Okay, so you're going to use that one?
		<b>Adam:</b> False, right?
	712	<b>Aaron:</b> Okay, yes, you could do that. I thought that you ought to look at an example. Because we use theorem 3.26 twice. So that's just..
Analyse	714	<b>Adam:</b> And the last, the last, it sort of follows from the argumentation we had before. That if it's not uniformly convergent, well then we don't know if $M_n$ just gets bigger and bigger.
	716	
		<b>Aaron:</b> Yes, we don't know what will happen when..but then we should prove that..well, here you have directly that it's only true when..here you don't know if it's true.
	718	
	720	<b>Adam:</b> No, you're right.

Aaron summarises their reasoning (not based on the sequence introduced by Adam) (lines 671-673, lines 677-683, and lines 685-686), but he only argues that each of the functions in the sequence and the limit function are bounded. He does not show how to find or construct a number,  $M$ , which bounds all the functions. Adam summarises the solution based on the number sequence he constructed  $\{M_n\}$  (lines 689-691, lines 693-698, and lines 700-702). He does not justify that the number sequence converges to the maximum of the limit function (which it does).

Plan		<b>Adam:</b> But you could show it just by looking at some sequence of functions.
	722	We could in fact take a very simple sequence, I think. What if we take $n$ of $x$ , then it's rather obvious..

Implement

724 **Aaron:**  $n$  of  $x$ ? You mean  $n$  times  $x$ ?

726 **Adam:** Yes, exactly. Then  $n$  gets bigger and bigger. Then you know that you have some interval, then maximum will increase all the time, because  $x$  becomes bigger and bigger.

728 **Aaron:** Yes, yes, that's obvious.

**Adam:** Then you have basically proved it.

730 **Aaron:** If it does not converge uniformly. Of course we have to start by seeing if it satisfies those conditions for  $f_n$ . Yes, because, obviously  $n$

732 times  $x$ , that's a continuous function, that's fair enough.

**Adam:** The question is, it's not supposed to be uniformly convergent. Well,

734 it can't be.

**Aaron:** No, but we have to show..we have to look past that uniform convergence.  $f_n$  is just a sequence of continuous functions and then the question is if it's uniformly bounded. And of course this  $nx$ , it's a sequence. Well, look, it's just what you said.  $nx$  is a sequence of continuous functions, cf. those rules of calculation we had in the beginning

736 about theorems for continuous functions, so this one is..this one is a sequence of continuous functions, but it's not uniformly bounded, because if we let  $n$  tend to infinity then this will tend to infinity for some fixed  $x$ , right?

738

740

742

744 **Adam:** Yes.

**Aaron:** Or for all  $x$  so far. And then we're done.

746 **Adam:** But, no, no. We are practically done, but it's not quite right, I think. Verify

748 Because what we have to show is not that it's uniformly bounded. We have to show that it's not because of uniform..because it's not uniformly convergent.

750 **Aaron:** Well, okay. But that's the part we sort of have omitted.

**Adam:** So what if we show that this one is not uniformly convergent?

752 **Aaron:** But that's what I am saying.

**Adam:** Okay, then I'll just shut up.

754 **Aaron:** Yes.

**Adam:** Can't we say that we're done?

756 **Aaron:** I don't know, I don't bother anymore.

Adam justifies the first supplementary request by referring to remark 3.27 (see appendix C.2), which states that the Extreme Value Theorem is wrong if either boundedness or closeness is removed from the statement. Concluding that since a result used in the proof is invalid this makes the statement untrue is an unsatisfactory answer. It is not possible to know for sure whether the statement in the task could have been proved without using the Extreme Value Theorem, so arguing that this theorem is not true if the interval is not closed and bounded (which is actually not even true!) is not enough. It is necessary to provide an example or construct a specific proof. The students justify the second supplementary request by introducing an example of a non-uniformly convergent sequence, which is not uniformly bounded, and argue that the statement is not true in this case.



### 6.3.2.1 Summary

What can the analysis from the perspective of structure and details reveal about the students' solution process?

By listing the conditions and comparing them with the definition of uniform boundedness, the two students imagine that the proof will emerge 'automatically'. The definition and the different conditions constitute parts of the details of the proof. The components provide the structure of the proof, but they cannot be derived merely from the details. To identify the different components in the proof, the students must draw on conceptual knowledge. Adam manages to explain that they have to show that each of the functions in the sequence is bounded by the same number. This is the first component. It seems that he manages this step because he has solved other tasks involving bounded functions, and the definition reminds him of this.

The second and third components are more difficult to deal with. The students try to use the conditions, but focus on the details. They do not consider the main problem in showing that each function in the sequence is bounded. To consider this, the students must draw on conceptual knowledge about the role of infinity in relation to sequences, i.e. the concept image of a sequence of functions. Instead, they start from the conditions and try to figure out what results they can deduce from them. This strategy makes them able to get the third component right, but they miss the second component. When verifying their intermediate result, they know that something is wrong, but the verification of their solution is not based on mathematics, but actually rests on one of Polya's heuristic ideas (*"have you used the whole hypothesis?"*, see page 56). The process that follows shows that it is difficult for the students to figure out why and how to apply the condition of uniform convergence. The heuristic approach made them realise the inadequacy of their argumentation, but it did not lead to a way to complete the argumentation.

Throughout the process, the students focus on the details of the proof, and they try to construct the proof from the details. When they discuss the details, they do not remind themselves about where in the proof structure they are, which is why they have difficulties completing the proof. The analysis based on structure, components and details explains the students' difficulties during the solution process and also why it takes the students more than an hour to present an answer that they regard as a proof of the statement: they try to extract the structure and the components from the details.

### 6.3.3 Team B

The following section only presents parts of team B's solution process. The purpose is to illustrate the analysis from the perspective of structure, components and details only. The students' difficulties not discussed here will be addressed in the next section, section 6.4. In the following analysis, I refer to the time-line

representation included in appendix E. The two students' notes can be found in appendix F. Cutting off their sentences and using numerous pronouns, the two students talk in an at times indecipherable fashion. This complicates the interpretation, making it difficult to translate the transcription into readable English.

During the first study period, the two students discuss the task, what it involves, and what they have to show.

- 16 (Time 3:48)
- 18 **Benny** What does it mean that it's uniformly bounded? Is it that all these ... subsequences, are those bounded? (He addresses STO) Read
- 20 **STO:** Well, the first part of the task says what it means.
- 22 **Benny** Yes, well, okay.
- 24 **Bob** Well, we have to look at ... sequences of functions which look like this. (Bob shows Benny something he has written on his paper) And show that this is smaller than or equal to some  $M$ . Eh, this means, that if a sequence of functions is uniformly convergent (he writes something down), then this holds (Draws an arrow to the definition of uniform boundedness)...
- 26 (1 min. pause)
- 28 **Bob** Well, we can see that because, uniformly convergent sequence..and if it's because it's uniformly convergent, eh to some function  $f(x)$  resulted in this [Mmm]. Are we going from this to that up there? (They both look at the paper) No,  $f_n(x)$  must be smaller than or equal to  $f(x)$  if it's uniformly convergent. Is it on a bounded interval?  $a$  to  $b$ , then..maybe Weierstrass M-test.
- 32
- 34 **Benny** What about this? (Benny shows him a page in the textbook) Extreme Value Theorem.

In an attempt to focus on the first component, Benny tries to formulate what it means to be uniformly bounded and to provide the details. Instead of going along with what Benny has started, Bob reformulates the statement in the task, "*if a sequence of functions is uniformly convergent, then this (uniform boundedness) holds*". He omits both premises  $P_2$  and  $P_3$  and focuses only on  $P_1$ . Avoiding the two premises is a characteristic of the whole solution process.

The details of the first component are not clarified before Bob moves on to find a result they both think they can use (line 32). He shows that his concept image of a convergent sequence of functions implies that the limit function is greater than all the functions in the sequence (line 31) (which is not correct).

After the first break, Benny reveals that he has in fact acquired a useful concept image of a uniformly convergent sequence in another mathematics course that he took the preceding year (lines 81-84):

- (time 11:33)
- 70 **Bob** A function converges to something ... uniformly. Explore
- 72 **Benny** I was wondering, what if you said that, limit supremum for  $n$  tending to infinity of  $f_n(x)$ ? Is that the same as  $f(x)$  which said that this was

- equal to capital  $M$ ? (Benny writes something down, while Bob looks at what he is writing)
- 74 **Bob** Well yes, it's also something like that I am working on, because  $f_n(x)$
- 76 converges uniformly to  $f(x)$ ..that's what I can't really figure out. We
- 78 have to look at..we have to show that this is smaller or equal to  $M$ ,
- 80 but it's a closed interval, so we just have to look at lim sup of either  $f_n(x) - f(x)$  for  $x$  in this interval or just at this limit. Eh, now it's just a general sequence of functions.
- 82 **Benny** It reminds me a bit about this other course we took. When we put a restriction on a function (Benny makes an interval about an imaginary function with his fingers, a strip around the function). Can't we do
- 84 that? To control how crazy it can be ...

Benny understands what Bob is talking about in his remark (lines 75-80), at least to a degree where it reminds him about some previously acquired image. The concept image introduced focuses the process on the second component. The third component is very briefly touched upon, but Bob's incorrect concept image disturbs the clarification:

- 100 (time 14:21)
- Analyse **Bob** Are continuous functions (He reads).
- 102 **Benny** Isn't it just it, then it's it?
- Bob** But it has something to do with how  $f_n(x)$  can move towards  $f(x)$ . It means that if it's under from the beginning, then it can't get to lie above, if.. $f_n(x)$  is smaller or equal to  $f(x)$  (Bob writes something down, while he speaks, Benny is following what Bob is writing) well at the point  $x$  ..
- 108 **Benny** That it's true all the way?
- Bob** Yes, and also at the limit. Limes  $f_n(x)$  smaller or equal to  $f_n$ . (Benny browses in the textbook) But that's not how uniform convergence (5-6 words un-detectable). Continuity, uniform.
- 110

After suggesting the use of and failing to construct a proof by contradiction, Bob wonders about the use of the property of continuity, premise  $P_2$ :

- 124 (time 17:20)
- Analyse **Bob** I just can't figure out what we are supposed to look at. We have to look at lim sup of some..of the functions and we have to make use of the fact that it's continuous to find out when it assumes it's greatest value, what the greatest value can be [Mmm]. Well, What I can't really find out is when this absolute value of  $f_n(x)$  assumes its, when it assumes its maximum value..that is the biggest value in this interval or if it..well, what we can use to show that it assumes its maximum value. If we can show that, then we have proven that this is smaller than this and then we have our  $M$ .
- 134 **Benny** What are you saying? That in this interval it has to assume its maximum value?
- 136 **Bob** Yes, that's  $M$ . That must be lim sup of this  $f_n(x)$  in the interval [Mmm], But this is where I can't figure out, because we don't have
- 138 any specific function, so I don't know when it assumes its maximum

- value, if it attains its maximum value when  $x$  attains its greatest value in the interval or..
- 140 **Benny** Can't you just choose..? Well, now that it's absolute value, can't you just choose an  $x_0$  in this interval?
- 142 **Bob** But we have to use that it's uniformly continuous or convergent so it tends to  $f(x)$  from below and that's what I'm thinking about..if we should use this fact that it's continuous then it must be true according to (non-detectable, three words), if  $f_n$  is smaller than or equal to  $f(x)$ . No, but that's, can't we just look under uniform convergence, and see what that, what that means? (They look it up) It's 7.2, and also 7.1. (They look at the book) Well, yes. If it's uniformly convergent then it's also where you can put an epsilon-strip around the function and then it should..the question is if you can just use  $f(x)$  because...
- 144
- 146
- 148
- 150

Bob tries to show that all the functions are smaller than a given number in one stroke. He thus tries to accomplish  $C_2$  and  $C_3$  simultaneously. In his argumentation he uses premise  $P_2$ , but he feels unable to use the property of continuity when the functions  $f_n$  are not specified. He explains to Benny that he does not know "*if it attains its maximum value when  $x$  attains its greatest value in the interval*". Benny suggests that they just choose a value (line 141). He actually addresses the third component, but he is not able to provide the details (it is unclear how he uses the absolute value to justify that they can choose a point  $x_0$ ). Again, Bob uses his incorrect concept image of a converging sequence of functions when he suggests that they use the property of uniform convergence to say that the sequence "*tends to  $f(x)$  from below*". He gets so confused that he finally (after twenty minutes) looks up the definition of uniform convergence (line 147). Now they both have the concept image of a uniformly convergent sequence as a sequence which can be confined to an epsilon-strip around the limit function. Bob seems to propose that they use the limit function to provide the number  $M$ , and by doing so he addresses  $C_4$ , but the correct details are missing. Bob continues:

- (time 21:09)
- 156 **Bob** Well, the  $f_n(x)$  lie inside, because it's uniformly continuous. If we then look at a suitably large  $n$  then it will lie in some neighbourhood around  $f(x)$ , an epsilon-neighbourhood. Analyse
- 158
- (Benny writes something down, Bob browses)
- 160 **Bob** We have to take this definition as our point of departure, well. It's like we have to find a..there must be a..we have to assume that they are uniformly convergent..this here. (He frames something on his paper)
- 162 It can at the most be.. $f_n$  of.. Damn it, no, it lies, well  $f_n(x)$  can fly around on each side of..
- 164
- Benny** Yes, but it can never cross this..
- 166 **Bob** Of  $f(x)$ .
- Benny** Wasn't it what you..well, then this must be some kind of  $M$ , in this interval?
- 168
- Bob** But, when is it that  $f_n(x)$  lies in this epsilon-neighbourhood? It does for some  $n$  bigger or equal to?
- 170

172 **Benny** Yes, for a capital  $N$  belonging to the natural numbers and then we  
choose an  $n$  bigger than. And then it's true for all  $x$  in our interval.

174 **Bob** We can't construct this proof, because we don't have a specific function.

Bob realises that they cannot use the limit function to provide a common number,  $M$  (line 163). Even though Benny tries to explain that this is not a problem, Bob finds it difficult to see how they can fix it. To him, all his troubles are connected to the fact that they do not have a concrete sequence of functions (line 173). Thus, even though they have an illustration of uniform convergence, it does not provide them, or at least not Bob, with a sufficiently useful concept image.

After thirty minutes when the implementation period begins, Benny tries to provide the details of the second component:

216 (time 28:55)  
Read **Bob** Where did you put the task? If there exists a number  $M$ ..  
Analyse 218 **Benny** In the interval there could be..well it could lie above..but is that the  
220 same as if it's..because it could be smaller than or equal to when you  
had this epsilon radius.  
**Bob** This, there exists, given an  $M$  (he might be saying  $n$ ) then all this you  
222 have done is true.  
**Benny** Mmm. So you better write that.  
224 **Bob** Yes. (Bob begins to write something down)  
**Benny** No, first let epsilon be given.  
226 **Bob** Yes, given an epsilon, there exists..  
**Benny** Then you choose an  $N$ , right?  
228 **Bob** Yes. Given an epsilon, there exists an  $N$ ..  
**Benny** Such that..it has to belong to the natural numbers.  
230 **Bob** The natural..then we have this down here. Can we erase that?  
**Benny** So  $n$  (he takes over the writing) bigger than or equal to  $N$  then for  
232 all  $x$  in the interval, right?  
**Bob** Yes. What I don't like about this is that we don't include anything  
234 about..what the interval looks like. This is just..it's just something we  
throw in at the end, closed interval from  $a$  to  $b$ , but in principle, it  
236 could be anything. But I think that we have to have some kind of  
maximum consideration and then say ..

The formalism is clearly not just something the two students can provide immediately. Bob is concerned with the fact that they have not used premise  $P_3$ , but he has a feeling about what they might use it for ("*a maximum consideration*", line 236). In the exchange that follows (not shown), he explains that the limit function might 'explode' if they do not look at a closed and bounded interval. So he clarifies a detail in the components  $C_2$  and  $C_3$ , but without realising it.

While Bob has been concentrating on the purpose of the closed and bounded interval, Benny has become confused about the presence of the sign ' $\leq$ ' in the statement in the task. He feels that he is able to show the inequality  $|f_n(x)| < M$ , but he thinks that the equal sign is only possible if they allow  $|f_n(x) - f(x)|$  to be

smaller than *and equal to* epsilon, but that contradicts the definition of uniform convergence:

- 416 (time 57:57)
- 418 **Bob** We are confused about why we have smaller than on our paper, but Verify  
 eh ..and our..what we have found is apparently independent of which  
 interval we are in. I am not quite sure what it is..I think we have found  
 420 something. We know that it's uniformly convergent and we know what  
 a uniformly convergent sequence looks like. It satisfies this one.
- 422 **Benny** And we also know, that..
- 424 **Bob** And we know that it's a closed interval, but we don't know what we  
 should use that information for.
- 426 **Benny** And we know that it's uniformly convergent on this interval.
- 428 **Bob** Yes, exactly.
- 430 **Benny** And then it must work.
- 432 **Bob** The problem is..
- 434 **Benny** We can have  $f_n(x)$  smaller than or equal to  $M$ , if we disregard that  
 you have to assume  $|f_n(x) - f(x)|$  smaller than or equal to epsilon,  
 then it works.
- 436 **Bob** So we have  $\epsilon + f(x) \geq f_n(x)$ ?
- 438 **Benny** Yes, and we put  $\epsilon + f(x)$  equal to  $M$ .
- 440 **STO:** So you think that you miss the equality sign?
- 442 **Benny** Yes, but we don't know what they use as argumentation in this  
 theorem 2.8. There they have equal to. There they say..this is for  $f_n$   
 minus what it converges to,  $a$ , is smaller than or equal to epsilon. And  
 444 what I..when you look at this..where you force  $f_n(x)$ , no not force, but,  
 no but it lies inside an epsilon radius of  $f(x)$ , then you have to in some  
 way provide argumentation that it can't..well, that it can be equal to.
- 446 **STO:** But what you have to prove, to show that its uniformly bounded, you  
 have to..what did you have to?
- Bob** We have to show that  $f_n(x)$  is smaller than or equal to some number  
 $M$  which is bigger than zero.
- STO:** If it's just smaller than some fixed  $M$  that could be enough, right?
- Smaller than *or* equal to.
- Benny** Well, okay.

The students are trying to reach the inequality in the statement by symbolic manipulation on the inequality coming from the definition of uniform convergence,  $|f_n(x) - f(x)| < \epsilon$ . Without being entirely certain about why it is 'legal', Benny removes the absolute value and obtains the result,  $\epsilon + f(x) > f_n(x)$ , and sets  $M = \epsilon + f(x)$  (neither of them realise that  $M$ , in this case, depends on  $x$ ). Now they get confused. Since the definition of uniform convergence only provides ' $<$ '. After inspecting the task they see that they need ' $\leq$ ' and this makes them doubt their reasoning. Had the students not overlooked the third component, this discrepancy would (presumably) not have caused trouble.

### 6.3.3.1 Summary

The analysis shows that the two students in team B do not pay attention to or let themselves be guided by the structure. They touch upon the first component

in the beginning, but they are unable to provide the details. Not having clarified that they need to show that *all* the functions in the sequence are bounded is partly responsible for the fact that they overlook all the functions in the head of the sequence.

Another factor responsible for overlooking the functions in the head of the sequence is Bob's (incorrect) concept image of a converging sequence of functions, as a sequence where  $f_n(x) < f(x)$  for all  $x$ . The illustration of a uniformly convergent sequence that they find in the textbook is in conflict with this concept image, because he can see that some of the functions,  $f_n$ , could lie above the limit function. Bob realises this, but it does not seem to affect his concept image, and they do not reconsider the functions in the head. The selected excerpts clearly show that both students confuse the different notions related to sequences of numbers, sequences of functions, and series. They have not acquired a solid concept image of a sequence of functions or of the uniform convergence of the aforementioned.

A huge part of the solution process focuses on the second component. They know from the illustration of a uniformly convergent sequence that the functions, for  $n \geq N$ , are confined within an epsilon-strip around the limit function, but they are unable to provide a complete set of details for this component. They know they have to use all the information provided in the task, but they are unable to make proper use of the condition that the interval is closed and bounded (and they never discuss the condition that all  $f_n$  are continuous). The reason for this could be that they do not recognise the necessity of having a closed and bounded interval and continuous<sup>2</sup> functions, because these conditions are incorporated in the illustration without explicit mention.

The students try to construct the details by symbolic manipulations without maintaining an understanding of the meaning of the symbols. Because of a discrepancy between the symbols used in the result of their manipulations and the statement in the task, they doubt their result. I assert that had they let the structure guide them, and thus realised the necessity of the third component, their confusion about the discrepancy would have disappeared.

In summary, an incorrect concept image and an illustration where the prerequisites are implicit prevent the two students from providing the details of the second component. Failing to focus on the structure results in holes in their justification, and these holes give rise to confusion about the details that they actually do manage to provide (although with some lack of rigour).

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<sup>2</sup> It is, as mentioned, enough to demand the boundedness of the functions to prove the statement in the task.

## 6.4 Main solving difficulties – including the solution processes of teams C and D

This section provides a summary of the main difficulties and common characteristics of the four teams' solution processes in the main study. Since the previous part of this chapter mainly deals with the investigation of the proposed hypothesis concerning structure, components and details, this section does not concern this particular difficulty. The different characteristics are illustrated with excerpts from the solution processes of teams C and D, and references are made to the excerpts already presented from the processes of teams A and B.

### 6.4.1 Lack of necessary resources and rich concept images

The four time-line representations clearly show that the four solution processes were quite different and probably did not have the same degree of success in their outcomes. Teams A, B and C all felt that they managed to provide a proof of the statement in the task. As described earlier, team A had a few unresolved issues, while team B had several. Team C was very far from reaching a correct answer, and team D gave up. I attempted to direct both teams C and D towards a correct answer.

In all four teams, the exploration periods are characterised by directionless searches in the textbook, where the students try to find results resembling the task. In teams B, C, and D this activity leads the solution process astray, while, due to similarities between the symbolism used in the task and convergence theorems for series (the letter  $M$  appearing in both cases), they mistakenly regard the sequence in the task as a series. Team B's process is not, in a crucial way, influenced by this misinterpretation.

The reason that periods of directionless exploration dominate the solution processes of teams C and D could be lack of necessary resources. As was the case with the processes of teams A and B in the pilot study, the students in teams C and D in the main study also find it very difficult to make sense of the task situation. Basically, they have not acquired any concept images of a sequence of functions or of uniform convergence, and when they finally look up the formal definitions, they are not able to reason on this ground. Since they do not possess the necessary resources to select useful results in the textbook, they are highly influenced by what they run into during their directionless searches in the textbook.

The processes of teams A and B are the only processes which contain periods of implementation. Searches in the textbook also occur during these periods, but in this case the searches are not directionless. During preceding planning periods, the students have identified properties of the theorems they are looking for. The difference between directionless textbook searches and searches where students have an idea of what they are looking for are conspicuous when it comes to the outcome of the search. When the students search without having specified what



they are looking for, they are unable to decide when a result might be useful. The following excerpt from team C illustrates this. It takes place forty minutes into the solution process, in the middle of an exploration period. The students are browsing through the textbook. Carl trips over definition 7.7, which in fact the two students had mentioned twenty minutes earlier even though neither of them appear to have any recollection of it.

- 428           **Carl:** Okay. (He starts to read from the textbook) If  $f$  is continuous at  
                   some  $x$ , 7.7, well, if  $E$  is a non-empty subspace of  $\mathbb{R}$  and we assume  
                   that  $f_n$  tends to  $f$  uniformly on  $E$ , we know that it does [Mmm] eh,  
 430           if every  $f_n$  is continuous at some  $x_0$ , then  $f$  is continuous at the same  
                    $x_0$ . (He reads from the textbook) Then we know that  $f$  is continuous  
 432           at  $x_0$ . So now you know that. (He laughs)  
                   **Carrie:** Okay. (The students turn a page in the textbook and look at theo-  
 434           rem 7.10, 15 sec. pause)

Carl figures out that the limit function is continuous, but because they do not have a proof strategy in mind, the result is completely useless to them. In the best of cases, the directionless searches in periods of exploration are only a waste of time, but in the worst of cases, the directionless searches actually contribute to an enhancement of the students' confusion and reduce their chances of clarifying the task. The observations contain several examples of such cases.

The students in teams A and B do not lack resources, at least not to the same extent as the students in teams C and D. As mentioned in the analyses of the two teams' solution processes, three of the four students (Adam is the exception) in teams A and B are not actually familiar with the notion of a sequence of functions or that of uniform convergence, so their concept images are not rich. Nevertheless, during the hour that they work on the task, they are able to develop some concept image of the notions, either by using their imaginations ("*That's sort of a sequence of sequences*") or by looking up the formal definition and seeing an illustration of it (the case with the notion of uniform convergence and the illustration with the epsilon-strip).

Since all eight students participated in the preparation study, it is possible to check how much time they spent preparing for the lesson in which the professor presented the notion of a sequence of functions (lesson 19 in the preparation study). Benny, Carrie and Carl did not do any reading before lessons 18-22 (Benny and Carl spent time on task solving in this period; Benny spent five hours on this before lesson 19 and two hours before lesson 21, while Carl spent thirty minutes before lesson 21). Dennis did not prepare at all for lessons 19-23, while Dan, Aaron, and Bob increased their preparation time during that period.

#### 6.4.2 'All information needs to be used'

Checking whether all of the information provided in a task has been used to reach a solution is often a decidedly useful strategy. Both teams A and B came so far

in their justification that they were able to use this heuristic idea to verify their solutions (examples can be found on page 221, line 256 and page 235, line 418).

The processes clearly show, however, that although this heuristic idea works very well to discover possible holes in the justification, it does not – at least in these two cases – distinctly lead to a way to remedy the holes (and in the event that all the information has been used, it does not guarantee correctness of the justification).

### 6.4.3 A need for specificity

Both teams A and B found it difficult to reason on general entities (see, for instance, page 215, line 63 and page 234, line 173). Examples can also be found in the solution processes of teams C and D. The following excerpt is taken from the process of team D. After forty-four minutes Dennis manages to provide an analysis of the task that could have got them on the right track, but he is confused about the fact that they do not have an expression for the limit function  $f$  (during his utterance he refers to the sequence of partial sums,  $s_n(x)$ , used in the definition of a convergent series):

216        **Dennis:** I just have..we ought to be able to choose something with..this  
 means that we can choose an  $N$  independent of  $x$ , is that right? Well,  
 218        just some capital  $N$ , right? Then we know that if we keep to this strip  
 (points at the epsilon-strip in the illustration of uniform convergence  
 in the textbook). Then we must..we have..even though we don't have  
 220        an expression for  $f(x)$ , well, that doesn't matter, does it? If we put,  
 for instance, some..if we take some..the question is if this, if this  $N$  we  
 222        choose..no, we can choose an  $s_n$ . That's  $f_n$ . Can't we say something  
 about this? Can't we replace this with some number? That's what I  
 224        am thinking about, because..

**Dan:** Well, because it says so here, okay, so we get this sequence.

At an earlier point Dennis expresses difficulties introducing a limit function,  $f$ , from the description that ‘the sequence  $f_n$  converges’. Since the task does not include a function,  $f$ , it is as if he is not allowed to assume that one exists. Again, this indicates that he uses the strategy ‘identification of similarities’ (in the sense of Lithner) to solve the task, and that he takes it very literally that he has to be able to find all the same symbols in the task as in the theorems he is trying to apply.

### 6.4.4 Symbolic manipulation and combination of results

The need for specificities are related to a belief that solutions emerge through symbolic manipulations. At some point or another, all four teams express the idea that a solution needs to involve symbolic manipulations and that a solution is achieved by directly combining known results.

The students in team A talk about adding “*some mathematics*” and “*theorem harvesting*”, while especially Benny in team B tries to reach the desired inequality

( $|f_n(x)| \leq M$ ) by manipulating the known distance between  $f_n$  and  $f$  ( $|f_n(x) - f(x)| \leq \epsilon$ ).

In team C, the students also find that proofs are constructed through symbolic manipulations. The next excerpt illustrates that they try to get from one expression to another without having understood what the symbolism stands for. They are trying to apply the Cauchy criterion (theorem 7.11 in appendix C.2):

**Carl:** It's this Cauchy-criterion.

440 **Carrie:** Yes. It's also something about, for all epsilon bigger than zero,  
there exists an  $N$  in the natural numbers such that  $n, m$  bigger than  
442  $N$  implies that..then we have like a limit-ish thing here. Or whatever it  
is. I just don't know..I can't figure out, how we get from this difference  
444 and then to the absolute value of this sequence.

**Carl:** Mmm. What's  $f_m$  here? (He looks in the textbook)

Almost near the end of their solving attempt, Carl describes how he usually solves tasks. To solve a task is not a question of having understood the task and the solution, it is about finding a result and using it. The excerpt begins at a point where Carl has just provided his justification of the statement in the task based on the Cauchy criterion. Carrie is unable to understand completely what he has done:

552 **Carrie:** I don't feel that..I can see that..I am not following like a hundred  
percent.

**Carl:** But that's what I am saying. ... You find some law which says that  
554 it's true and then you try to prove it using that law.

Carl indicates that it is not important to understand; they just need to use some already proven results. Dennis in team D explains the need to construct the proof through calculations in this way:

200 **Dennis:** I see it like this, I want to find..we have to have some..it's about  
isolating this  $f_n$ ..I want to reach an expression, but I don't quite know  
202 if it's even correct. I think so. But it's just to like choose something,  
eh, that can correspond to  $M$ . That's what I'm thinking. I think I am  
204 a bit confused.

**STO:** So you want to use the definition of uniform convergence and then in  
206 some way do some calculations?

**Dennis:** Yes.

208 **STO:** Have I understood you correctly?

**Dennis:** Yes.

Dennis would like to obtain an expression he can do calculations on or manipulate to reach the desired inequality. This might not always be a bad strategy, but since the team has not acquired an understanding of the task, because of a lack of necessary resources and inadequate concept images of the concepts involved, it is not possible to use such a strategy correctly.

These two beliefs – that solutions involve symbolic manipulations and include other results – are not necessarily wrong and might not lead to solving difficulties. It seems, however, that some teams more than others believe that a solution can be achieved *merely* by manipulating the symbols, and that a solution can be constructed *merely* by combining textbook results selected only on the basis of an accordance between the symbols or notions used in the task and those used in the theorems. Understanding the theorems and the task is not perceived as necessary. Hence, this perceived *sociomathematical norm of proof production* can prevent the students from trying to develop mathematical understanding.



## 7 Discussion and conclusions

“There is no empirical method without speculative concepts and systems; and there is no speculative thinking whose concepts do not reveal, on closer investigation, the empirical material from which they stem.” (Albert Einstein)

The presentation and discussion of the research findings are structured according to the two research questions. Hereafter I discuss the strength and range of the findings, including the influence of the research design on the findings. The chapter ends with some reflections about the didactical usefulness of the framework developed for the structure, components and details.

### 7.1 What are the main difficulties university students experience when trying to justify mathematical statements in tasks?

The sixteen solution processes observed in the pilot study, and the four processes observed in the main study provide the data material for the conclusions put forward in this section.

The previous chapter ended with a summary of the main characteristics and difficulties (aside from the identified difficulty relating to the lack of clarity about structure and details) of the four solution processes observed in the main study. Combined with the findings from the pilot study related to student’ solving difficulties presented in chapter 4, the following list of different types of solving difficulties provides a compact answer to the first research question. The order of the list resembles the order in which the difficulties will appear in a prototypical solution process. The difficulty that has occupied the most attention in this dissertation is placed last.

1. Lack of relevant mathematical resources;
2. Lack of ability to construct meaning of new mathematical situations without guidance;
3. Establishment of an unfortunate sociomathematical norm of proof production;
4. Difficulties in determining a solving strategy;

5. Inadequate concept images;
6. Directionless textbook searches enhance student confusion and possible misconceptions of the task;
7. Need for specificity and symbolic manipulations.
8. Lack of clarity and ability to distinguish between the structure and the details of a proof.

The different types of difficulties and characteristics have not received the same investigative attention in this dissertation, and they are not believed to carry the same degree of responsibility for the students' solving difficulties. In the following, each of these items will be summarised, and their interdependence will be discussed.

### **7.1.1 Lack of relevant resources**

When students meet a mathematical task they have an already established basis of mathematical knowledge that they are perhaps able to activate and use to solve the task. This is the knowledge that Schoenfeld [1985] calls a person's mathematical resources.

Almost all the students in the eight teams observed (pilot study and main study) at some point or another revealed that their knowledge base/resources were insufficient in relation to solving the tasks. In two of the teams (teams A and B in the pilot study), the students had not acquired all the necessary prerequisite knowledge addressed in the course. The students in teams C and D in the main study were unfamiliar with many of the central concept definitions involved in the task and were unable to separate properly between different related concepts such as function sequences and series. Even Aaron in team A in the main study revealed that he was not familiar with the concept of a sequence of functions when initially faced with the task (he was able to create a concept image of the concept in the beginning of the process so his insufficient knowledge base only had a minor impact on the process).

It comes as no surprise that students experience difficulties solving mathematical problems when they have not put in the effort to learn the relevant concepts and notions so they at least know the formal definitions and have some idea about how to apply them. What is surprising – and deserves further investigation – about this finding is that these students are studying at university level, they have chosen to base their professional careers on mathematics, and they have already passed three or four elementary university courses in mathematics. At this point in their education they have clearly not learned how to study in an appropriate and constructive way.

### **7.1.2 Inability to create meaning**

The solution processes revealed that the students were unable or found it very difficult to create meaning of new mathematical situations without being guided

by an authority such as the professor. The weaker students who had inadequate resources could not use the task to develop a conceptual understanding of the concepts and notions involved.

This finding implies that for this group of students, unguided problem solving will most likely not result in an enhancement of their conceptual understanding. When students are not able to create meaning of new mathematical situations, they can only hope to be able to solve routine tasks or they have to rely on superficial strategies when the tasks are not routine tasks. This leads to the conclusion that some students need more guidance and help in the stage where meaning of a new mathematical situation is created.

### 7.1.3 Unfortunate norm of proof production

Several of the solution processes revealed that an unfortunate sociomathematical norm of proof production had been established among the students. The norm entailed that proofs of mathematical statements could (and should) be constructed through the following process: Key notions from the task should be used to identify relevant theorems, and by combining the wordings of the theorems, the proof of the statement would reveal itself more or less automatically. Understanding the task and the identified theorems do not enter as a prerequisite.

This was not only a characteristic feature of those solution processes where the students lacked the necessary mathematical resources. Aaron and Adam in team A in the main study talked about writing down the definitions and the conditions in the task and then seeing if the justification of the statement would (magically) appear. This sociomathematical norm is thus not a mere consequence of inadequate mathematical resources.

The nature of some tasks demanding the justification of a statement entails that the proof in fact can be constructed by a combination of the wordings of a couple of theorems. It is possible to use this strategy successfully without having acquired a full-fledged conceptual understanding of the notions involved in either the task or the selected theorems.<sup>1</sup> The solving strategy might lead to a correct answer, but it does not require nor enhance the students' conceptual understanding of the concepts and notions involved. A focus on content and conceptual understanding is not required nor necessary and might even be considered a waste of time. This norm indicates that the students have misunderstood what it takes to learn mathematics.

There is a subtle difference between how a professional mathematician would justify a statement and how a student with this described norm acts. Both of them would try to use and combine definitions and already proven results to justify the statement. The difference is that the professional mathematician bases the selection of useful results on his or her understanding, and not on superfi-

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<sup>1</sup> An observed, although undocumented, solving episode in course A in the preliminary study confirms this.



cial comparisons between the task and the theorems. Nor does the professional mathematician think that the justification would appear ‘miraculously’ just by comparing and combining the wordings. It is thus very likely that professors would not realise that this norm has been established, and would thus not be able to correct the students’ misconception.

#### **7.1.4 Difficulties in determining a solving strategy**

Even though many of the students showed signs of the described sociomathematical norm of proof production it is not the case that their solution process can be explained and understood only by reference to this norm.

Some of the students tried and were able to develop an understanding of the task. Again, team A in the main study can be given as an example. But even though these students to some degree were able to construct the meaning of the task and of the concepts involved, they still had difficulties finding a solving strategy. They found it difficult to get a good idea about how to construct the proof. This difficulty seems to be related to the belief found among the students that it was necessary to use tricks/good ideas to construct a proof. They did not see an immediately relation between the statement to be proven and these tricks/good ideas. It seemed as if they thought they could achieve the ability to get a good idea just by seeing a lot of task solutions and without reflecting on each case about the relation between the statement and the proof idea.

#### **7.1.5 Inadequate concept images**

The solution processes verify that students, also at university level, build and use insufficient and wrong concept images. Furthermore, the observations reveal that wrong concept images can lead students to create wrong concept definition images. This could explain the observation that the students hesitated to look up the formal definition. When the students thought they had developed a correct image of a definition, it looks as if they could not see the need to look up the formal definition. Their hesitance might also be related to their belief that looking in the textbook is a sign of failure and something that they should avoid (team A in the pilot study). In the solution processes when they – for a lack of other options – eventually did check the definitions, they found that they could not use the definition to make progress. This supports their conception that the formal concept definition is less important or less useful compared to their concept image of it. It would be interesting to examine further whether this tendency to rely more on developed concept images than on the formal definitions exists in other student populations and in different teaching practices as well as what the reasons could be for this behaviour.

Comparing solution processes where students have developed a wrong image of a concept definition (pilot study, task 1) for their processes when they have not yet developed an especially strong concept image (pilot study, task 4) showed,

perhaps surprisingly, that faced with more unfamiliar notions (e.g. a sigma algebra), the students were able to choose a solving strategy based on the definition much faster. It thus seems worse (in relation to finding a successful solution strategy) to have acquired an insufficient or wrong concept image than not to have acquired a (or only a very sparse) concept image at all. One reason for this could be that students who have not developed any concept image of a given concept knows that they only have the formal definition to rely on. The task becomes a 'game' with very clear and simple 'rules'.

It should be noted, however, that the two particular definitions (the definition of continuity and the definition of a sigma-algebra) have very different levels of complexity. To check if the definition of continuity is satisfied demands the mastering of epsilon-delta argumentation, whereas the definition of a sigma-algebra (only) implies simple set-theoretical considerations. This difference in the level of complexity could have an influence on the difference in the students' behaviours in these two cases.

#### 7.1.6 Directionless searches in the textbook

In almost all of the solution processes exploration periods appeared when the students browsed through the textbook without having decided in advance what they were looking for. The fact that these episodes exist and have this characteristic seem to be related to the established sociomathematical norm of proof production. The students look for theorems containing the same concepts or notions as in the task and where a connection between them is offered.

Lithner [2003] saw the same kind of behaviour among undergraduate calculus students. Students used 'identification of similarities' to reason when solving tasks, and although they did not manage to solve the task this way, some students were actually able to progress.

The students in my study were not able to use this strategy successfully, because they were unable to identify and chose relevant theorems when they had not articulated the nature of the result they were looking for in advance.

The time spent doing textbook searches was also used to look for examples to copy. This strategy is useful at lower levels of mathematics where the focus is on the development of technical skills, and where the examples show how to use particular techniques. But at an advanced undergraduate level many of the examples in textbooks function to set the boundary for a certain concept or to show abnormalities or warn the students not to rely too unreflectively on their mathematical intuition. The examples are seldom of a general nature nor are they directly applicable when constructing new proofs. Therefore, the students have a hard time locating examples, which would provide them with a solving strategy.

### 7.1.7 Need for specificity and symbolic manipulations

Even at an advanced level, the students still find it difficult to deal with general tasks where no explicit algebraic/analytic expressions appeared. Some of the students in this study explicitly stated that they could not make any progress due to the lack of explicit expressions of the involved notions. This finding seems to link to a belief that a solution should be reached through, and hence contain, symbolic manipulations. Many of the students thus showed clear signs of having developed symbolic proof schemes.

The focus on making calculations on specific expressions, and the need to include symbolic manipulations are likely to be caused by the students' difficulties getting a good idea for the proof, and the misconception that proofs can be constructed without developing and using concept images of the notions involved. When they have not developed sufficient concept images, and they do not possess conceptual understanding of the concepts, it does not come as a surprise that they try to apply the tools they are used to applying to reach a solution, namely to perform calculations of some sort.

### 7.1.8 Lack of distinction between proof structure and details

The main part of this dissertation has revolved around the promotion and examination of a hypothesis concerning the observed solving difficulties. In order to examine the hypothesis a framework defining what structure, components and details of a proof are has been developed. For many of the teams in the pilot study and the main study it seemed that their difficulties and confusion could be explained by their inability to handle the dialectic relationship between the structure and details of a proof. The lack of ability to distinguish between structure and details became especially clear in cases where the students possessed the necessary mathematical resources, but had not developed sufficiently rich concept images of the concepts involved.

The students focused a great deal on constructing the proof by supplying the proof details, and not by paying explicit attention to the proof structure. The processes showed that it was difficult for the students to gain a sense of understanding of the task only by focusing on the details, and they were not convinced about the accuracy of the details they did manage to provide because they did not see how the details were related to the structure (e.g. pilot study, task 2 and main study, team A). Not realising the proof structure and the relevance of the details in relation to the structure made it difficult for the students to verify the correctness of their solution.

The developed framework, which is used to examine the hypothesis, can also provide one explanation as to why mathematical analysis is not an easy subject for students to learn. Six analyses of proofs have been provided, and they all reveal a huge complexity in the proofs of even very simple statements. It is not a trivial task to uncover the complex hierarchy of components and details within the

structure. The details are of course connected to the components, but often the details of one component depend upon or refer to the details of other components in the structure. On top of that, the nature of the details differ tremendously. The realisation of some components just involves simple observations, while other details entail technical calculations, demand challenging constructions, and the ability to juggle with and negate definitions and statements involving multiple quantifiers. Students should be able to handle this complexity while they try to acquire the concepts of mathematical analysis and develop rich concept images that are not erroneous.

## 7.2 Are these difficulties related to the way the students have been taught?

It is obvious that students who have received the same teaching do not necessarily behave in the same way. They have perhaps not acquired the mathematics to the same degree and they do not solve tasks in the same way.

A teaching practice and a teacher/professor do not control the students' learning outcome, but the teaching practice sets a crucial stage for learning. It promotes, influences and supports student learning through specific learning activities and other more subtle signals, but it does not carry the sole responsibility for how the students carry out these activities nor if and in what ways they receive the signals. Some students perceive the signals as they were intended by the professor, others misunderstand or fail to hear them.

Students possess different abilities for learning mathematics. They prefer and are comfortable with different teaching styles. Some students like having dialogues where they feel free to pose questions and where the tone is more relaxed, while others – for different reasons – prefer the professor to carry out lectures as ready-made talks given at conferences. Creating the ideal learning opportunities for each individual student is both impossible in theory and in praxis.

One danger connected to a study concerning how student difficulties are related to the way the students have been taught is that the study can end up concluding that the difficulties are the result of bad teaching. In order for the outcome of an examination of the research questions to be interesting, it is necessary that the teachers can be described as good teachers. Good mathematics teachers are passionate about what they do and they take their jobs seriously; they reflect on their teaching goals and the means for achieving them. Good teachers respect their students, and they do not patronise them when they experience learning difficulties. Instead, they try to understand the reasons behind their difficulties. All the teachers in this PhD study are good teachers who give it their utmost to help the students learn the trade of mathematics.

This research study thus documents the obvious: even good teachers cannot prevent students from experiencing difficulties when learning mathematics. But

precisely because the professors can be characterised as good teachers makes it possible to detect and understand difficulties of a more general nature, and to examine how these difficulties connect to intended and unintended signals from the teaching practice.

In the following, I present and discuss how and to what extent the already presented solving difficulties and the teaching practice (main study) might be related.

### 7.2.1 The focus on details

The identified lack of attention or ability to distinguish between structure and details observed in the solution processes were also to be found through an analysis of the ways proofs are validated in the classroom.

Analysing proof validations in the classroom from the perspective of structure, components and details showed that these situations lacked clarity about how a proof strategy was found and how the proof structure emerged from it. One of the examples provided showed that the professor assumed the students could construct the proof structure once they realised the need to apply the definition of continuity (the proof validation starting on page 129).

The professor took for granted that the students were well-acquainted with the proof structure so he did not need to draw much attention to it. He focused on explaining the details. The data material revealed that the students found it difficult to understand the details, because it often was not clear to them which proof structure and thus which components the details were a part of and which ones they referred to.

The student focus on details and their perception that proofs can be constructed merely through symbolic manipulations, and a combination of theorems (without conceptual understanding) seem to be connected. The analysis of the teaching practice from the perspective of proof schemes showed that some of the elements and unintended signals in the teaching practice could give rise to the development of symbolic proof schemes, which could explain the students' behaviour and misconceptions. The professor tried to establish the sociomathematical norm of discussing mathematicians in *spe.* Had this norm in fact been established in the classroom, it might have enhanced the students' opportunities for developing more adequate beliefs about justification and proof construction. But the students were unable to act in a way that could help establish this norm. A reason for this might be found in their way of preparing, which will be discussed in the next section.

### 7.2.2 Lack of sufficient preparation

The lack of sufficient resources observed in the two teams in the pilot study was to some extent caused by an insufficient conceptual understanding of the prerequisite material for the course. The students' lack of sufficient resources

might possibly be related to previous the teaching practices they have been the recipients of, but this can of course not be concluded based on my data.

For two of the teams in the main study, the processes were highly affected by the students' inadequate resources. Because of the preparation logs it was possible to examine how much time these students had spent on course preparation. In the last part of the course when sequences of functions were introduced, three out of four of the students on the two teams did not spend any time reading and only short periods of time on task solving because of project deadlines. The students, however, did attend the lectures. From their solution processes it was clear that only attending lectures without preparing for them did not provide the students with sufficient understanding of the concepts introduced.

Is the student lack of preparation related to the teaching practice? In order to answer this question, I will invoke the course in the supplementary study. One of the professor's goals (in the supplementary study) was to make the students read the textbook, and in order to reach this goal, he implemented tools of assessment (online assignments) that ensured that the students read the textbook before each lecture. Since the course in the main study had a pass/fail final exam, and there were no assessment demands during the course, it is not surprising that the students, when, for instance, project deadlines approach, assigned a lower priority to course preparation. I must emphasise that I am not judging which of the two teaching practices is the better. I am only pointing to a possible connection between the student lack of preparation and the assessment requirements of the course in question.

The lack of preparation could be an unintended result of the established social norm for preparation. In practice, the professor did not expect the students to spend many hours reading a chapter and solving associated tasks before class, and he did not orchestrate the teaching practice in a way that required the students to prepare themselves for the lectures. This means that it was possible and actually acceptable for the students to attend a lecture without having prepared for it at all. If it turned out that none of the students were able to participate in a discussion or dialogue initiated by the professor, he would still carry out the planned presentation of the textbook chapter – but without active student participation. Thus, even though the professor did talk to the students about the importance of preparation, the students might have received a different signal from the way the lectures were carried out.

Besides revealing that some of the students did not spend enough time on preparation, the preparation study also revealed that the students did not prepare in a manner that would enable them to participate in meta-mathematical discussions about the range and limitations of the concepts introduced, and the mathematical structure of axioms, definitions and theorems. They read the textbook linearly and this inevitably led to a focus on the details of the proofs.

### 7.2.3 Developing concept images and the focus on concept definitions

The graphical time-line representations of the lessons in the course in the main study clearly demonstrated that time devoted to issues specifically enhancing conceptual understanding was limited and often came in smaller periods of time during other teaching activities, such as proof validation or as a transition between two activities where the professor erased the blackboard. Proof validations occupied often longer, more coherent periods of time as the course progressed. While it might not be easy for students to expand their concept images through proof validation, this aspect of the teaching practice could explain why some of the students developed limited concept images of the new concepts introduced in the course, and were not able to correct wrong concept images.

One ‘solution’ could then be to include more periods of time with the explicit aim to discuss the concept definitions and to develop concept images. The data material verifies that this is not a straightforward task. In the solution process of team A in the main study, Aaron revealed that he was unfamiliar with the concept of function sequences. The recordings of the professor’s introduction of this concept reveal that Aaron participated actively in a discussion about the difference between pointwise and uniform convergence, which was characterised as ‘concepts and mathematical structure’. For me, it was impossible to detect that Aaron had in fact not realised what kind of object he was discussing the convergence of. This episode shows that having what appears as meaningful mathematical discussions about the properties of concepts does not guarantee the development of concept images.

The finding that some of the students hesitated to check the formal concept definition, and instead relied on their developed concept definition image, could be related to misinterpreted signals from the teaching practice. One of the examples provided of a proof validation situation (the excerpt on page 169) shows that the students were not asked directly to look up the concept definition if they were unable to repeat it. Another example (the excerpt starting on page 129) shows that the professor did not explicitly say that he used the formal definition of continuity. He referred to it as ‘the more operational version’ of the high school definition of continuity in terms of limits. It seems as if the professor tries to make the students realise the mathematics on their own instead of telling them what is happening. This is a feature of his objective to suppress the development of authoritarian proof schemes. But there are also examples where the professor explicitly said that they have to be able ‘to recite the definition of the limit in their sleep’. Aaron (in the main study) revealed that he had in fact heard what the professor said, but in spite of that he had not managed to memorise the definition (see page 215, lines 68-70).

It seems that especially the mathematically weak students in the pilot study and in the main study misinterpreted the lack of attention devoted to definitions and did not fully realise the importance and role of definitions in mathematics in general, and in proof construction in particular.

Compared to the teaching practice in the supplementary study there seems to be a difference between how students in Denmark and in Canada behave. Many of the students in the supplementary study were able to recite definitions when the professor asked, whereas this was not the case in the main study. This difference might be explainable by looking at the different teaching traditions in the two countries. In Denmark, a major shift in the teaching tradition in the 1970s occurred in which teaching activities connected with rote learning (including memorising procedures and facts) were more or less abandoned. So even though the professor in the main study says that the students should be able to recite the definitions in their sleep, the Danish teaching tradition over the last thirty years could be the reason why the students did not make an effort to memorise the definitions. In the supplementary study the assessment requirements were focused highly on memorisation, and I speculate that Canadian primary and secondary schools, to a much larger extent, emphasise the role of memorising. If and to what extent this is a reasonable explanation needs to be investigated.

#### 7.2.4 Getting a good idea

Getting a good idea (or seeing the main point of the task or statement) is very important in a proof construction situation and could be viewed as the most striking difference between proof validation activities and proof construction activities. But getting a good idea is related to the statement and not something that ‘falls down from the sky’. If every proof a student sees or manages to construct with the help from a professor is based on what is perceived as a trick or a good idea where the relation to the statement remains hidden, the student will never be able to construct proofs without guidance.

The establishment of the sociomathematical norm among the students that task solutions or proof constructions demand the use of tricks and good ideas with no apparent relation to the task formulation, indicates that the teaching practice did not succeed in enabling the students to identify the connection between the statement and the proof strategy. This does not mean, however, that the professor did not mention it at all. Examples from proof validations in class have been provided (see example on page 175) where the professor explains why he chooses a certain proof strategy, for instance, a strategy based on the definition of continuity. These incidents have in common that the professor mentions the strategy in a way that sends the signal that it is immediately comprehensible or even trivial. He thus does not indicate that the identification of a connection between the statement and the strategy is a point worth attending to. The students are, as described, more inclined to participate actively, for example, by posing questions when more concrete issues are addressed, such as the justification of the details in the proof. How this tendency is related to the way they have prepared has already been discussed.

Problem solving activities in a teaching practice would be a natural place to focus on the relation between the statement and the proof idea. In the teaching



practice observed (main study), task solving was treated through two different activities, partly during the professor's presentations of solutions to textbook tasks, and partly during the solving sessions. The presentation of the task solutions resembled the professor's presentations of textbook proofs. Attention was devoted to details, and the proof strategy and the resulting proof structure were often taken for granted. Solution strategies were not paid much explicit attention. When the students were given a smaller period of time to look at a task before the solution was discussed, they were able to participate more actively in the discussion of the solution, and the professor was then able to work with the establishment of the social norm of participation and the sociomathematical norm of accepted forms of argumentation.

The observation of the student-professor interactions in the solving sessions showed that the professor's function was mainly to check the students' solutions and to provide hints (often perceived as tricks) when the students were stuck. This was a behaviour that the students to a large extent reinforced. The excerpts provided showed that the professor did manage to make students aware of the importance of precision in the argumentation, but the student-professor interaction did not naturally lead to a discussion about the choice of solution strategies from a general point of view, nor to a discussion about the relation between the statement to be proven and the chosen proof strategy (mostly because the students were not concerned with this issue). Consequently, the task solution activities did not seem to enhance student awareness of the difference between proof structure and proof details nor make the students realise how a good idea behind a proof relates to the statement to be proven.

### 7.3 The research question originally intended

In the introduction I explained that the research questions formulated were a consequence of the data material. Since many of the students experienced some kind of difficulty solving the tasks, the empirical material did not provide a firm foundation for examining the more general research question originally intended.

Before exposing the students to the tasks I had hoped and expected that they would not be able to solve the tasks immediately since the tasks were designed with the intention of being problems and not routine tasks. Nonetheless, it came as a surprise that so many of the teams found it that difficult to solve the tasks or simply to make reasonable progress. All the tasks in the pilot study and in the main study can be solved by examining the relevant definitions, and invoking some main theorems.

Therefore, the research ended up focusing on the relation between students solving difficulties and the teaching practice, and not on a general characterisation of the way students solve tasks. The data material presented does, however, contain one example of a team which managed to reach a satisfactory solution during a solution process containing an expected amount of uncertainty and ex-

ploration. In team D in the pilot study, Danny had obviously developed sufficiently rich concept images of continuity and metric spaces, and possessed the necessary technical skills related to carrying out epsilon-delta arguments, such that he was able to transform what was a problem at first into a routine exercise. The proof structure became clear to him, and the rest was just a matter of providing the details.

## 7.4 Validity, reliability, and generalisability of findings

Has the research design turned out to be a tool by which it is possible to capture something significant? Are the findings of a substantial character? How strong is the foundation on which the findings rest?

The project and thus the findings concern something as transient, momentary and complex as teaching and learning and the relation between the two. The main research tool for data generation has been observations preserved in audio and video recordings. In the case of the teaching practice, the observations were structured around an observation template, which resulted in the identification of ten main categories. The time-line representations of the lessons as well as the analyses of the teaching practice from the perspective of social and sociomathematical norms, and proof schemes made it possible to detect signals that the teaching practice send to students. This could either be signals intended by the professor or unintended signals that may lead to misconceptions in the students. Using these tools and analytical approaches, I find that I have been able to provide an overview of the teaching practice, identify main elements characterising the teaching, and in general am able to see something significant that I could not see just by observing the teaching practice.

In the presentation of the characterisation of the teaching practice, the excerpts have been used to illustrate the findings resulting from the use of the different analytical tools, since this was the only possible option. This is not the case with the analysis of the solution processes. The analysis of the solution processes focuses on the mathematical content of the students' dialogue; in order to assure a high level of reliability (or trustworthiness) of the findings, I found it necessary that the findings were clearly substantiated by the transcripts. This explains the relevance of presenting entire solution processes, and not just a few excerpts illustrating my findings. Choosing a monographic presentation makes it possible to include enough transcripts so the reader does not have to 'take my word' for the existence of the empirical foundation. The reader is able to make his or her own judgement of the relation between the data and the findings, and this possibility enhances the reliability of the findings.

In the following I elaborate on the influence of the different parts of the research design on the research findings.

#### 7.4.1 The design of the solution process observations

In the methodological chapter I explain why I chose to design the observations of the students in the way that I did. It seems that the collaboration naturally leads to useful solving protocols with a minimal interference from the observer. The down side is that the students sometimes disrupt each others' line of thinking instead of generating synergy. After having observed the students solving the designed tasks, it is clear that especially the weaker students often obstruct each others' potentially constructive approaches. It thus appears that the research design could induce additional solving difficulties, which is a methodological downside that Schoenfeld [1985], the inventor of this particular research design, did not report or take into consideration.

It might be the case that some of the students would have been able to come further in the solution process had they worked alone, but since it is mostly the weaker students who confuse each other, it is very likely that they would not have been able to come much further on their own. The students in team A in the main study, for instance, did work alone for shorter periods of time and came up with two different solution attempts. In this case, the research design did not prevent individual work.

One solution could have been to modify the research design so the students were forced to work alone for a couple of minutes, and then asked to explain their possible progress to each other after each period. The possible drawback of this design is that one of the students could have solved the task, and then the solving protocol would not have recorded the solution process, but instead the other students' ability to understand a completed solution.

#### 7.4.2 The framework of structure, components, and details

Using the perspective of structure, components, and details to analyse a specific teaching activity, namely a proof validation situation, revealed a likely link between the teaching practice and the solving difficulties. But to what extent can I claim that the influence from the proof validation situations is the most significant? Would it not be more likely that the influence from problem solving activities in the teaching practice plays a much larger role in the students' difficulties than how proofs are validated in class?

A general answer might well be 'yes', but based on the two particular courses in the pilot study and the main study, I would have to answer 'not necessarily in these two cases'. I will explain why.

Out of the five courses I have observed during the PhD project, two of the courses contained tutorials where teaching assistants assisted the students when trying to solve tasks and also reviewed task solutions at the blackboard. Course B in the preliminary study, and the courses in the pilot study and in the main study did not contain sessions where the professor or a teaching assistant systematically reviewed task solutions in front of the whole class. In these courses, the

students used the solving sessions to solve tasks with the assistance of the professor/teaching assistant. The only activity which would provide the students with a coherent presentation and possible discussion of the justification of a mathematical statement would in fact be proof validation situations. Additionally, the students participating in the pilot study and in the main study all reported that it was very important for them to participate in the lectures, and many of the students valued the professor's presentation and found it necessary in their acquisition of the mathematical content. For that reason, I find it fair to claim that proof validation situations in class have a significant influence on the students' abilities to justify unknown statements.

### 7.4.3 The tool for characterising teaching practices

Inspired by the preliminary study, an observation template was devised before and developed during the pilot study and then tested on the course in the main study. This template combined with the graphical time-line representation tool developed by Schoenfeld, resulted in a tool for representing a lesson graphically. The developed tool was also tested on the course in the supplementary study. It was possible to use this tool to characterise other courses than the course it was developed to describe. This enhances the generalisability of the tool.

The time-line representations of the lessons are founded on the assumption that time spent on the different categories is the most significant factor in characterising a teaching practice. The time-line representation does not (explicitly) say anything about the quality of the teaching/learning activities taking place. Moreover, the information concerning how the dialogues between the professor and the students proceeded and what mathematical content was discussed cannot be extracted from the representations either. Due to these limitations the graphical representation tool cannot stand alone in a characterisation of a teaching practice. Therefore, the characterisation was supplemented with analyses based on the frameworks of classroom norms and of proof schemes.

### 7.4.4 The design of the solving session observations

It is difficult to conclude what effect the time spent on solving session lessons had on the students' justification of statements, because the professor did not address the whole class. The discussions that the professor had with one group of students naturally affected the way this particular group of students would solve tasks. In order to be able to draw substantial conclusions about how the dialogue with the professor affected the students, it would have been necessary to observe each groups' dialogue with the professor. This could have been done had I followed the professor around instead of staying with one particular group at each solving session. Unfortunately, this design would have made it impossible to record what the students were doing when the professor was not visiting their particular group.

#### 7.4.5 Generalisability of the findings

The findings presented originate from (a few) case studies. The question is then whether there is reason to believe that the findings can be replicated under similar, but naturally not identical, circumstances.

The main finding that students find it difficult to distinguish between the structure and details of a proof was observed in two different student populations attending two different courses, but at the same university and with the same professor. The finding is closely related to the nature of mathematical proof, and it is thus likely that the same type of difficulty could be found in other student populations. The specific character of the findings thus suggests a high generalisability, which could be further examined in other student populations.

The finding that the professor in the pilot study and in the main study took for granted that the students had realised the proof structure and only needed explanation of the details was also evident in the supplementary study. The fact that another professor at another university in another country behaves similarly suggests that this is not a finding that is dependent on the particular professor in the pilot and main study, nor is it a consequence of the specific setting of the two courses.

### 7.5 What scientific territory has been reclaimed?

How does this research study contribute to the research landscape of mathematical education? In light of the literature reviewed in chapter 2, this study clearly contributes by addressing a level of mathematics education (advanced undergraduate) and a mathematical subject (advanced mathematical analysis) which have not yet been receiving much attention. Apart from this contribution, the research study also identifies phenomena related to mathematics teaching and learning which have not been documented in the research literature before.

This research study focuses closely on the identification and further examination of the role of proof structure and proof details in the teaching and learning of mathematics at this level. The interplay between proof structure and details has of course been addressed in the literature since these are pivotal aspects of mathematics (see for instance [Leron, 1983]). In fact, the dialectical battle between proof structure and proof details could be viewed as an example of the ongoing discussion about what to focus on in mathematics education, for example, the development of conceptual knowledge and relational understanding or the enhancement of technical skills and thus focusing on the development of instrumental understanding. This study offers empirical material and a theoretical framework which can be used to structure, and deepen this discussion.

Many of the other solving difficulties observed have previously been documented in the literature in some way or another – although only rarely for students at this educational level. A large amount of literature documents that students find it difficult to deal with quantifiers, and the notion of limits, which

makes mathematical analysis difficult to learn. Tall and Vinner show that the notions of concept image and concept definition are important when describing certain types of difficulties that students encounter. Raman demonstrates that students have difficulties identifying a key idea that can lead to an idea of how to construct a proof. Lithner finds that calculus students tend to use reasoning based on the identification of similarities, making it difficult for them to cope with non-routine tasks, while Schoenfeld's studies show that student solution processes do not involve enough periods of evaluation and control such that periods of exploration can lead to 'wild goose chases'. All of these difficulties have also been documented in this study for the particular group of students at the particular educational level.

Some of the findings, however, contribute new information regarding the already identified set of difficulties. This study documents that students can develop incorrect concept definitions from inadequate concept images, and that an incorrect concept definition can prevent students from using the correct formal definitions. This is of course destructive for a solution process. In relation to this, the study also shows that the students focus and rely on their developed concept images at the expense of the formal concept definitions. It appears as if the formal concept definitions lose their importance once the students feel they have developed a concept image of it.

What appears to be a new discovery is the unfortunate sociomathematical norm of proof production that was established among the students in this study. This norm entails that students focus on superficial similarities between the task and already proven results; hence it can be seen as an extension of Lithner's notion of 'identification of similarities' when proofs are involved in task solving. Since the students' attention is shifted away from conceptual understanding, this norm also appears to have negative consequences for the students' abilities to learn mathematics.

This study also contributes with an examination of student study habits and how and to what extent their habits influence and are a consequence of the teaching practice. Because of the small number of students participating in the preparation survey, it is clear that more research is needed concerning the role and importance of preparation in relation to mathematics teaching and learning.

Finally, this study provides an 'existence proof' that learning mathematics up to this educational level is a struggle, and that there is a need to carry out more educational research, also at an advanced university level, in order to help students learn mathematics.

## 7.6 Pedagogical considerations

The pedagogical consequence of the observation that incorrect or inadequate concept images can lead to erroneous concept definitions is that the teaching should pay a great deal of attention to the development of correct and profound concept

images, especially regarding concepts the students have encountered before but in a different form, or when introducing generalisations of well-known concepts (e.g. the generalisation of number spaces to metric spaces) where there is a risk that characteristics of the specific case wrongly are carried over to the general case. The professor should be aware of these pitfalls and address them explicitly, and not take for granted that students easily, and without developing misconceptions, can develop new concept images and adjust the old ones.

With regard to the established norm of proof production, it is clear that professors must be careful not to send signals that unintentionally confirm the students' misconception of how proofs should be constructed. This is a difficult task since proof construction uses and combines definitions and already established results. If the student does not understand the theorem used or why using it is relevant, it is very likely that students will experience the professor's proof validation as a confirmation of their misconception.

I see using the proposed framework of structure, components and details as an option for avoiding this pitfall, since more attention to the proof structure could make the students focus more on conceptual understanding and less on trying to gain acquire mathematics through superficial comparisons. Because I have not examined the pedagogical usefulness of the framework in this study, the following observations are of course my personal reflections.

I find that the framework developed can be used for guidance and as a tool for student preparation in addition to being used as a tool for discussing proofs in the classroom. The preparation study indicates that students do not know any other way to read a traditional mathematical text except linearly. A linear approach does not result in a focus on the proof structure. If students were instead asked to detect the structure of the proofs, consider how this structure related to the chosen proof strategy, identify the components, and analyse how the details of the different components might be related, this would provide them with a very constructive tool for preparation and a solid basis for discussing the proofs in class.

The professor and some of the students participating in the main study verified that the framework could be a useful pedagogical tool. About half way through the development of the framework, I had the opportunity to give a seminar in which the professor and some of the students from the main study were also in attendance. After the presentation of the framework, which included examples of a proof validation situation in class as well as a solution process, I received responses from both the professor and the students. The students said that seeing the proof analysed according to structure, components and details, enhanced their sense of having understood the proof.

The professor found the proposed framework for analysing a proof validation situation to be sound. He explained that he in fact had intended to give the students a sense of both structure and details when reviewing a proof, but when reading the transcripts he realised that this message did not always appear as

clear as he had intended.

The responses from the students and the professor were limited and might not be representative. They have not been recorded in any way, and are just immediate responses to the presentation of the framework. However, they indicate that my research findings are found to be relevant and useful by those who are meant to eventually benefit from research in mathematics education.





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# A Course plan of the course in the main study

The course plan for the course as presented on the course web site. First column contains the date. Second column refers to chapters in the textbook [Wade, 2004]. Third column contains assigned tasks. Hand-in assignments are marked with bold. Underlined tasks indicate that a hint is available. The underlined hand-ins contain examples of students' answers to some of the hand-in assignments. I am responsible for the English translation.

<b>8.9</b>	Course beginning, <b>1.1</b> Ordered Field axioms	<b>1.1:</b> 1, <b>2</b> , 10, 9, 6, 7, 8, <b>5</b>
<b>12.9</b>	<b>1.2</b> The Well-ordering Principle. <b>1.3</b> The Completeness Axiom	<b>1.2:</b> 1, <b>2</b> , 3, 6, 9, 7 <b>1.3:</b> <b>1</b> , 3, 4, 6, 7, 8, 9, 10, <b>5</b>
<b>15.9</b>	<b>1.4</b> Functions, Countability and the Algebra of sets	Recapitulating tasks. <u>1st hand-in assign.</u>
<b>19.9</b>	<b>2.1</b> Limits of sequences	<b>1.4:</b> 1, 3, <b>5</b> , 9, 6, <b>11</b>
<b>22.9</b>	<b>2.2</b> Limit Theorems, + Exec. 2.2.10	<b>2.1:</b> 1, <b>2</b> , 7, 3, 4, <b>5</b> , 6, 8
<b>26.9</b>	<b>2.3</b> The Bolzano-Weierstrass Theorem <b>2.4</b> Cauchy sequences	<b>2.2:</b> <u>1</u> , <b>2</b> , <b>4</b> , <b>5</b> , <b>6</b> , <b>9</b> , 7
<b>29.9</b>	<b>2.5</b> Limits supremum and infimum	<b>2.3:</b> 1, 2, 3, 8, 11, <u><b>9</b></u> 3rd hand-in assign. <b>2.4:</b> 1, 2, 3
<b>3.10</b>	<b>3.1</b> Two-sided limits	<b>2.4:</b> 4, 7, 8, 9, <b>5</b> <b>2.5:</b> 1, 2, <b>3</b> , 7, 4, <b>9</b>
<b>6.10</b>	<b>3.2</b> One-sided limits and limits at infinity	<b>3.1:</b> 1, 2, <b>3</b> , 5, 7, <b>9</b> 4th hand-in assign.
<b>10.10</b>	<b>3.3</b> Continuity	<b>3.2:</b> 1, 2, <b>3</b> , 4, 7, 8, <b>9</b>
<b>13.10</b>	<b>3.4</b> Uniform continuity	<b>3.3:</b> 1, <b>2</b> , 4, 6, 7, <u>8</u> , 10, <b>9</b>
<b>17.10</b>	<b>4.1</b> The derivative, <b>4.2</b> Differentiability theorems	<b>3.4:</b> <b>1</b> , 2, 3, 4, 5, 7, 8, 9, <u><b>6</b></u>
<b>20.10</b>	Self study	<b>4.1:</b> 1, 2, 3, 6, 7, <u><b>4</b></u>
<b>24.10</b>	<b>4.3</b> The Mean Value Theorem	<b>4.2:</b> 1, 2, 3, 8, 9, <u>7</u>
<b>27.10</b>	<b>4.4</b> Monotone functions and the Inverse Function Theorem	<b>4.3:</b> 1, 2, 3, 5, 6, 7, 9, <b>4</b>
<b>31.10</b>	<b>5.1</b> The Riemann Integral	<b>4.4:</b> 1, 2, 3, 5, 7, 8, <b>4</b>
<b>3.11</b>	<b>5.2</b> Riemann sums	<b>5.1:</b> 1, <b>2</b> , 4, 5, <b>8</b> , 9, 7, 6, 3
<b>7.11</b>	<b>5.3</b> The Fundamental Theorem of Calculus	<b>5.2:</b> 1, <b>2</b> , 5, 6, <b>7</b> , 8
<b>10.11</b>	<b>5.4</b> Improper Riemann integration	<b>5.3:</b> 1, <b>3</b> , 4, <b>5</b> , 7, 8, 9, 10, 11, <u>8th hand-in assign.</u>
<b>14.11</b>	<b>5.5</b> Functions of bounded variation ( <i>cursory</i> )	<b>5.4:</b> 1, <b>2</b> , 3, 4, 5, 6, 7

<b>17.11</b>	<b>6.1</b> Introduction	<b>5.5:</b> 1,2 <b>5.6:</b> 1, 2, 3, 4, 5, 6, 7, 9th hand-in assign.
<b>21.11</b>	<b>6.2</b> Series with nonnegative terms	<b>6.1:</b> 1,2, 3, 4, 5, 6
<b>24.11</b>	<b>6.3</b> Absolute convergence	<b>6.2:</b> 1,2,3, 4, 5, 6, 7
<b>28.11</b>	<b>6.4</b> Alternating series, <b>6.5</b> Estimation of series, <b>6.6</b> Additional tests	<b>6.3:</b> 1, 2, 3, 4, 5, 6, 7, 8 <a href="#"><u>Hints</u></a>
<b>1.12</b>	<b>7.1</b> Uniform convergence of sequences	<b>6.4:</b> 1, 2, 3, 4, 5, 7 <b>6.5:</b> 1 <b>6.6:</b> 1
<b>5.12</b>	<b>7.2</b> Uniform convergence of series, <b>7.3</b> Power series	<b>7.1:</b> 1, 2, 3, 4, 5, 6, 8, 9
<b>8.12</b>	<b>7.3</b> Power series	<b>7.2:</b> 1, 2, 3, 4, 5, 7, 8
<b>12.12</b>	<b>7.4</b> Analytic functions	<b>7.3:</b> 1, 2, 3, 4, 5, 7
<b>15.12</b>	<b>7.5</b> Applications	<b>7.4:</b> 1, 2, 3, 4, 5, 7, 9
<b>19.12</b>	Recapitulation, Evaluation and Christmas goodies	<b>7.5:</b> 1, 3, 4, 5, 6

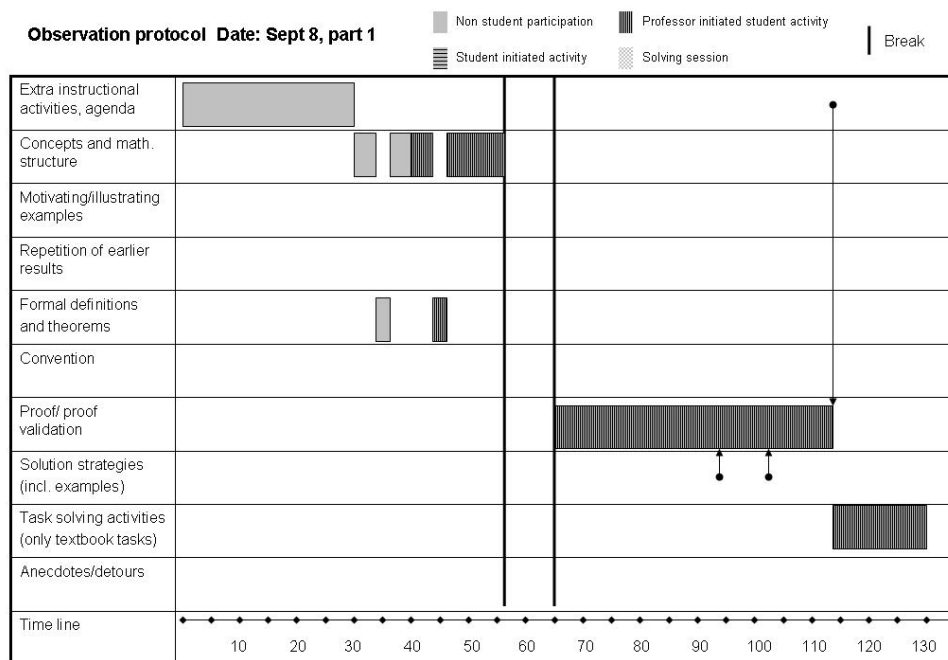
## B Time-line representations of lessons

This appendix contains time-line representations of twenty-five lessons observed in connection with the main study, and two lessons from the supplementary study.

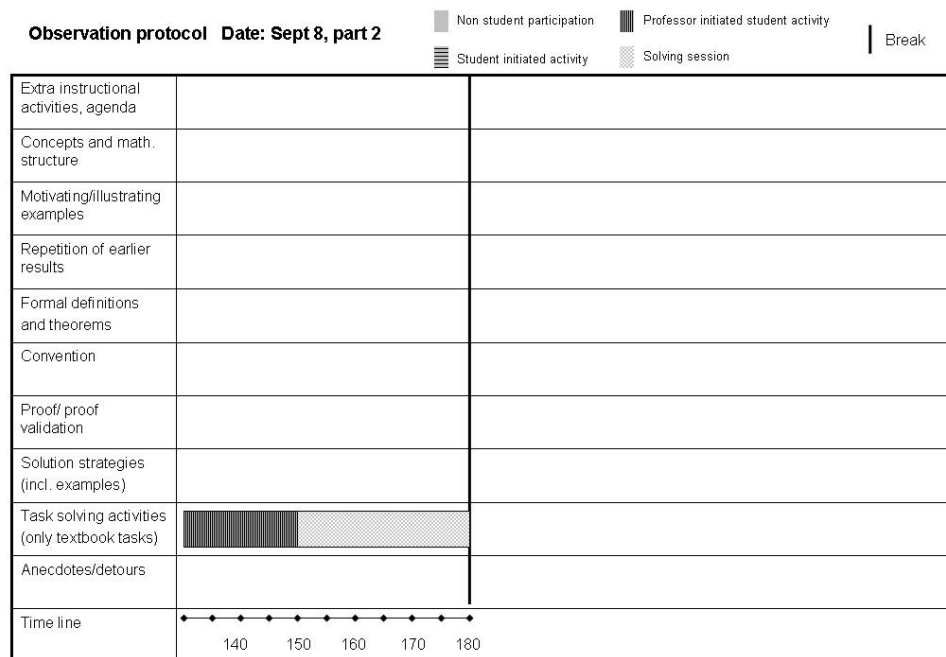
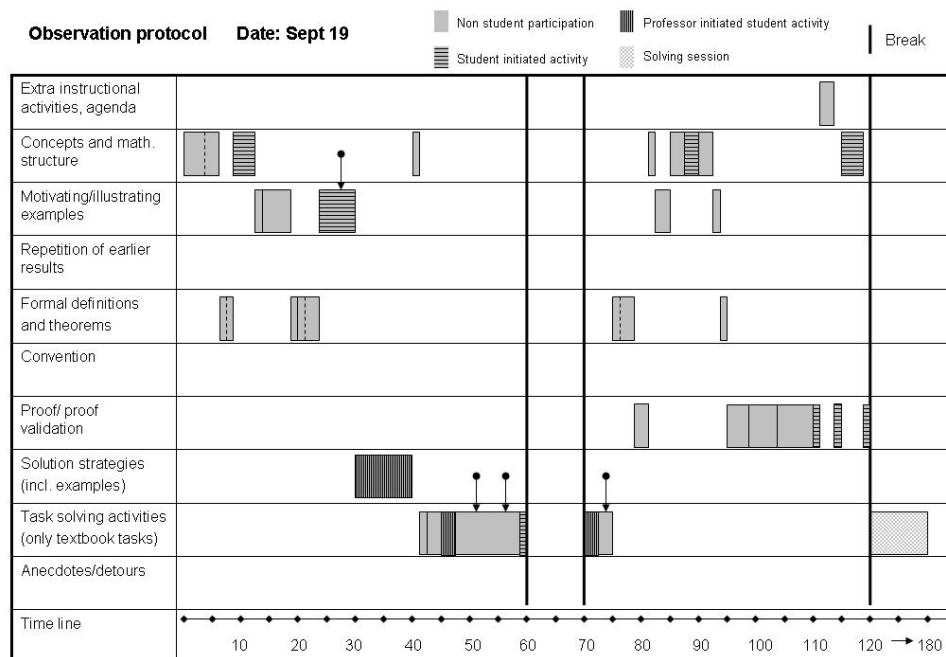
The lessons in the main study lasted three hours and usually the last hour was spent on task solving in small groups, where the professor was available for help. This part of the lesson is not represented in details (notice that the time-line jumps from 120 to 180 in most of the time-line representations).

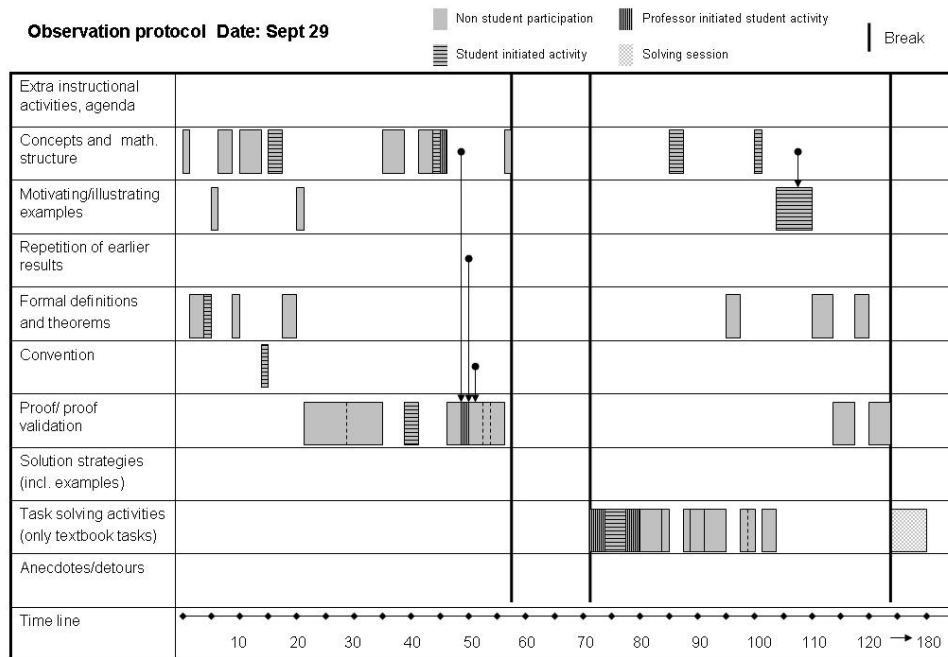
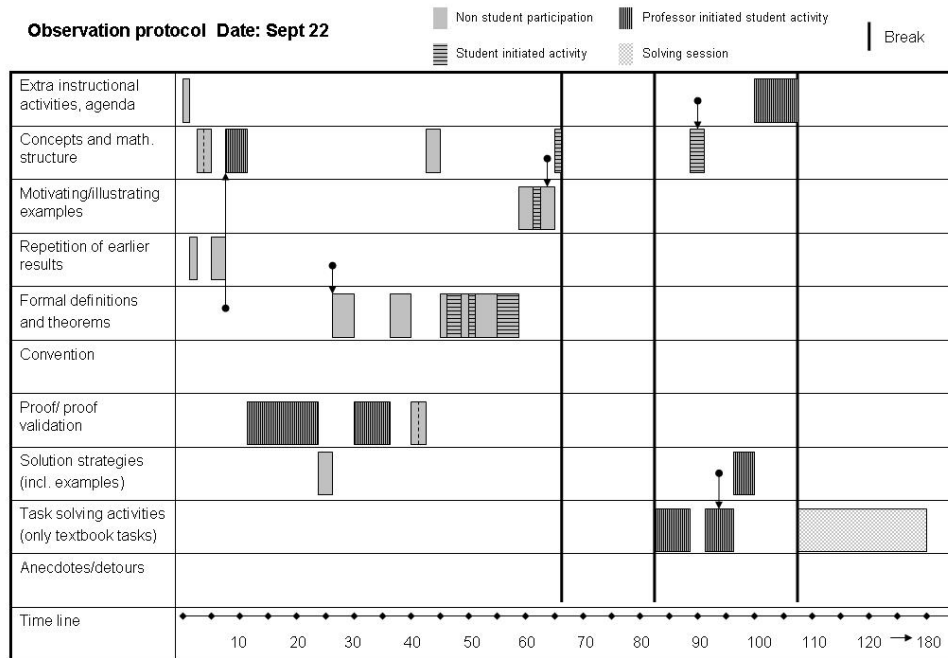
The course observed in the supplementary study was divided between 50 minutes lectures held by a professor and 50 minutes tutorials held by a teaching assistant.

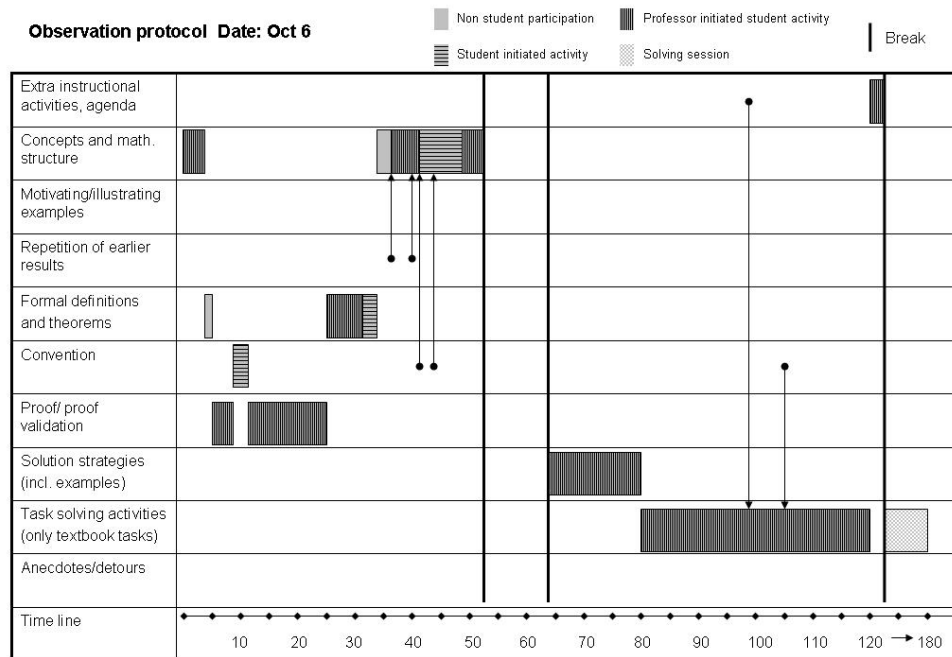
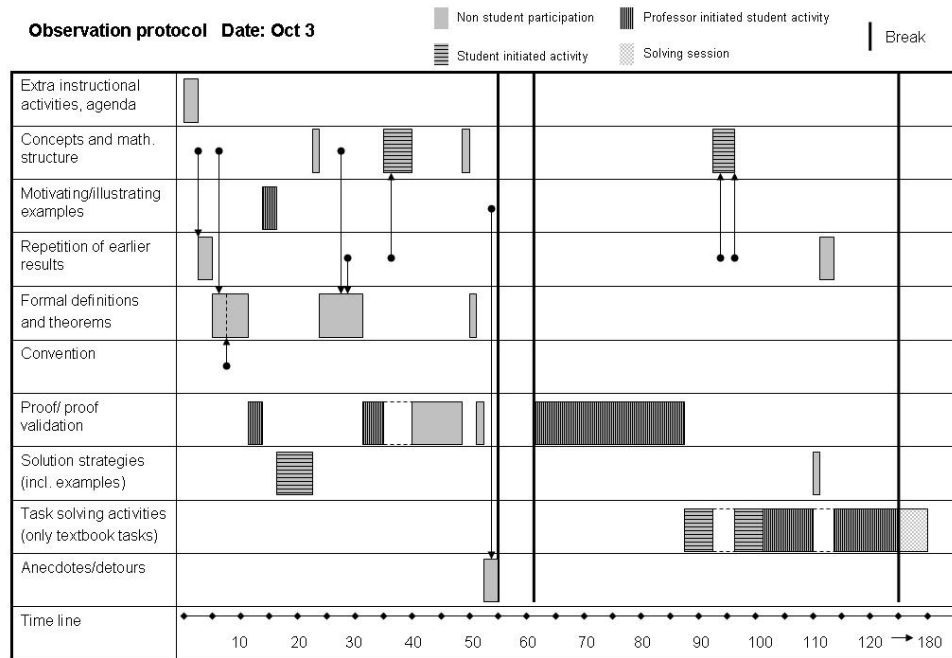
## B.1 Main study

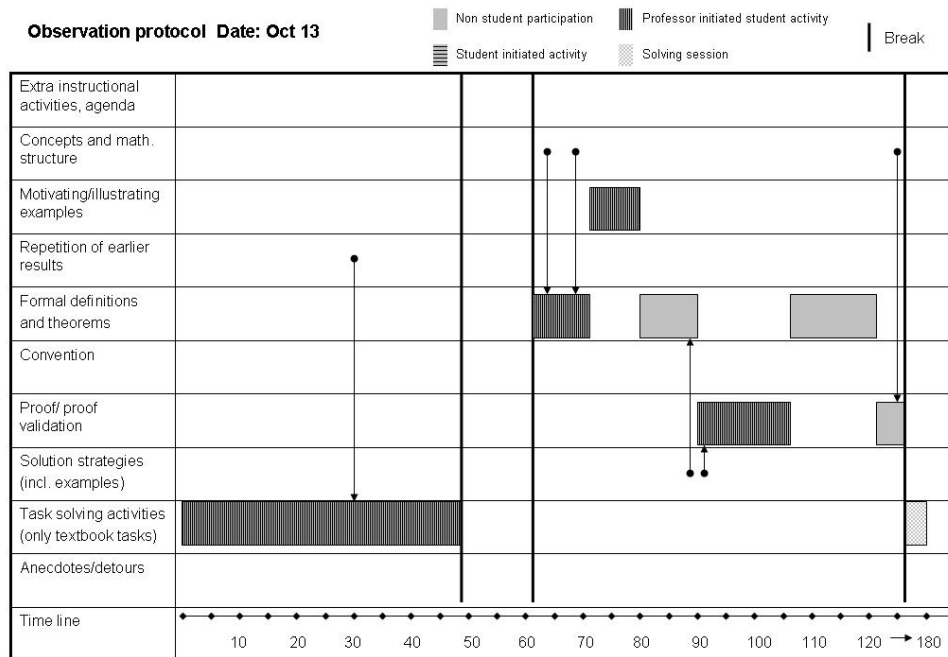
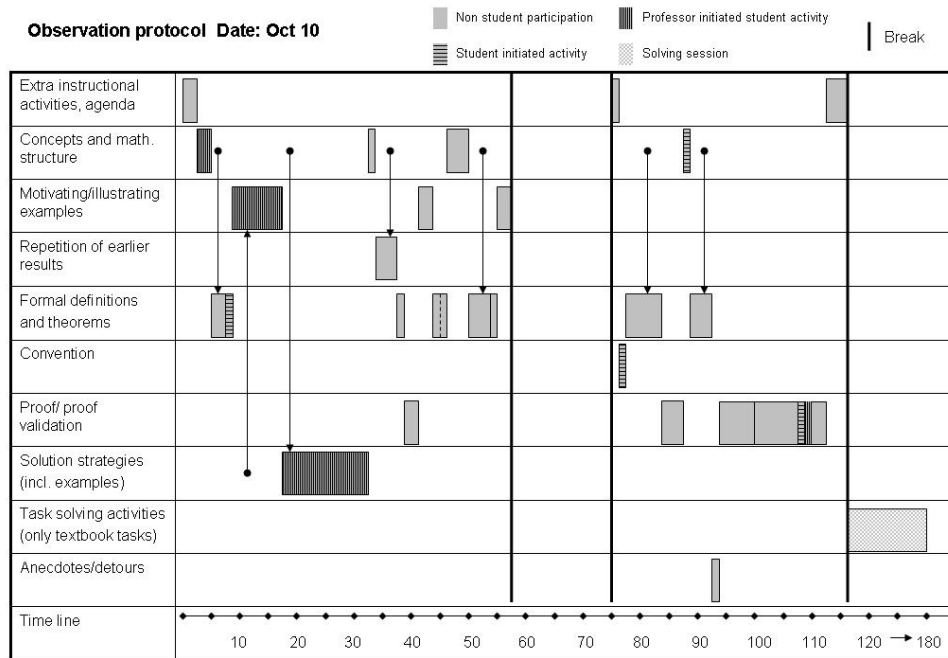


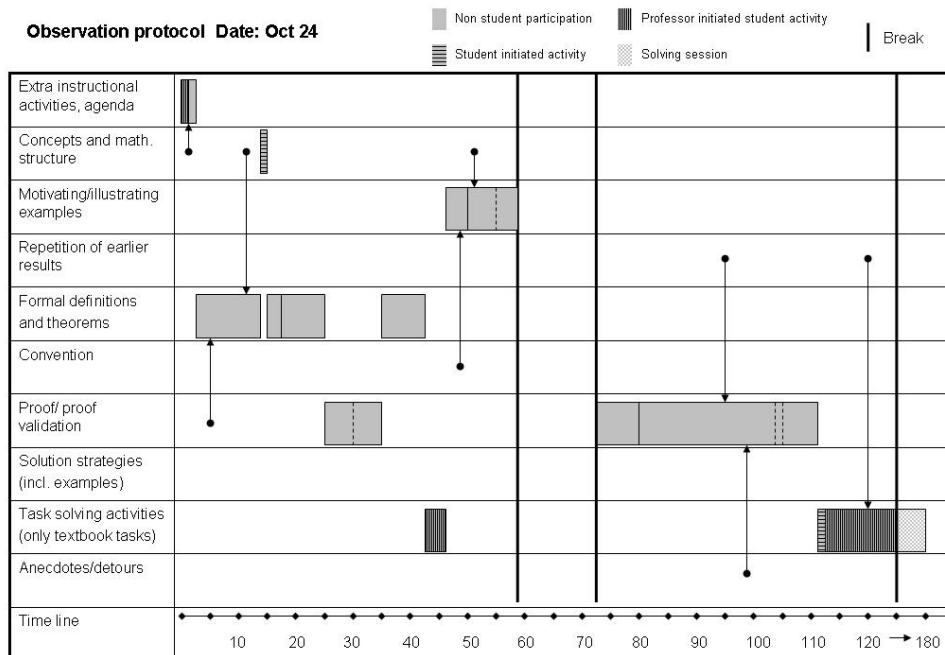
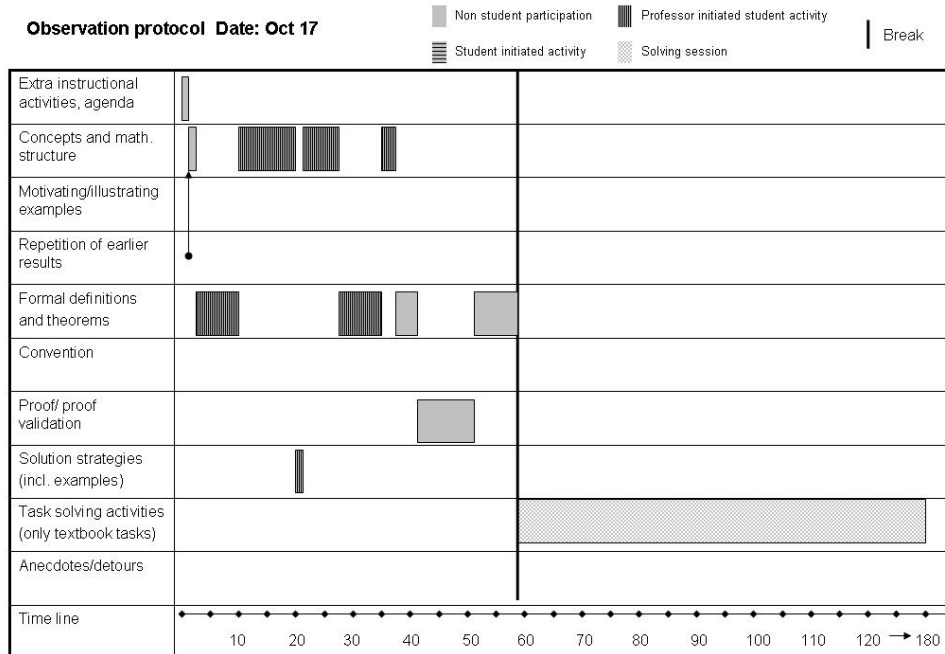


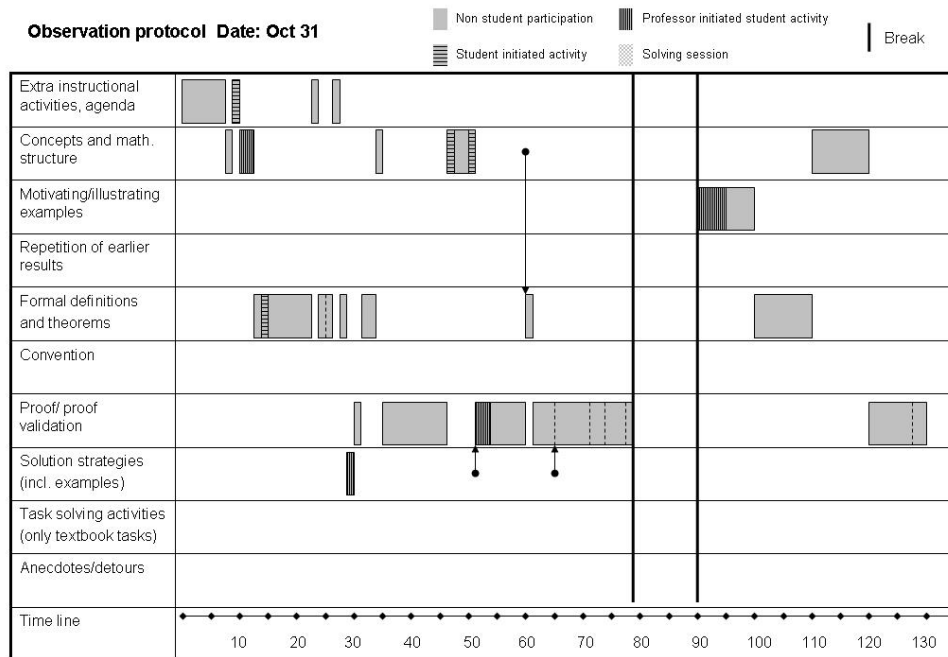
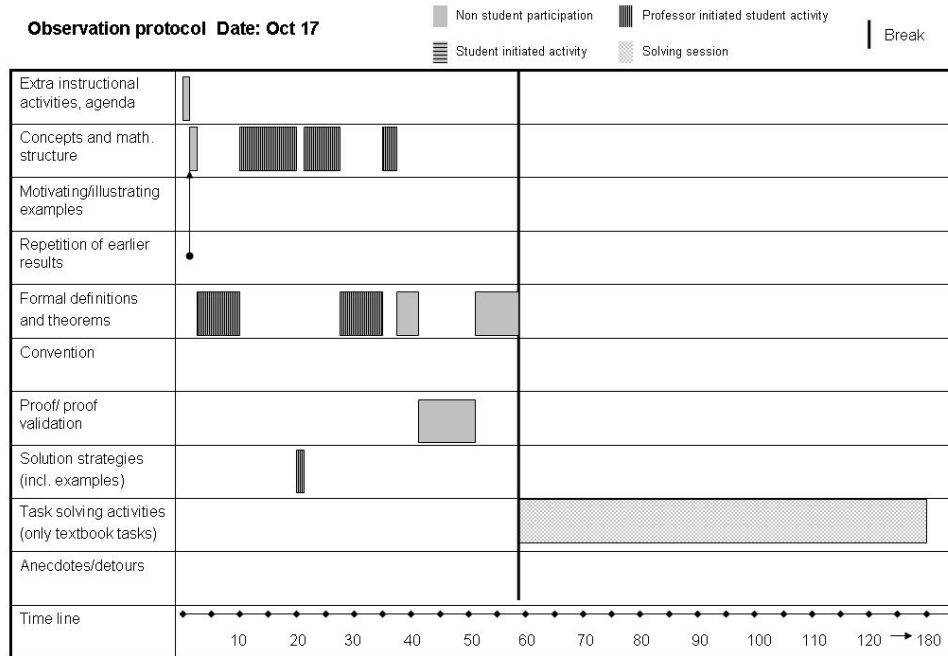
**Observation protocol** Date: Sept 8, part 2**Observation protocol** Date: Sept 19

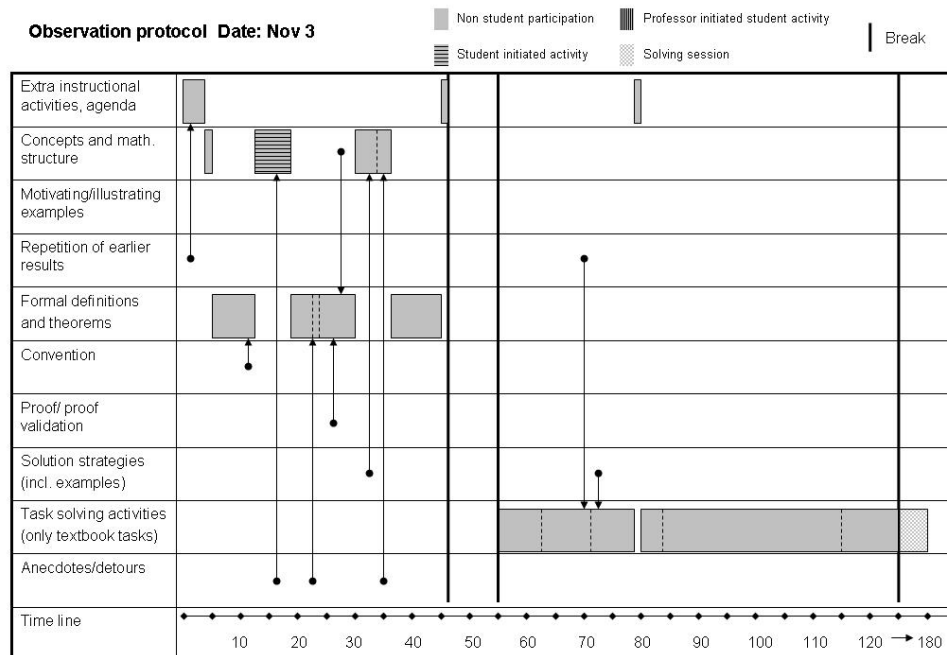
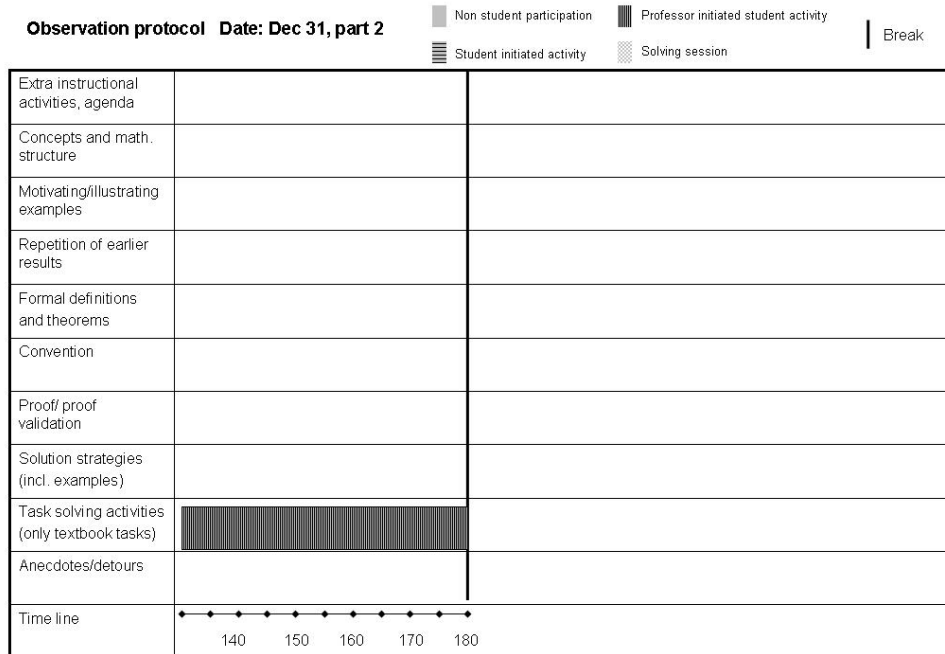


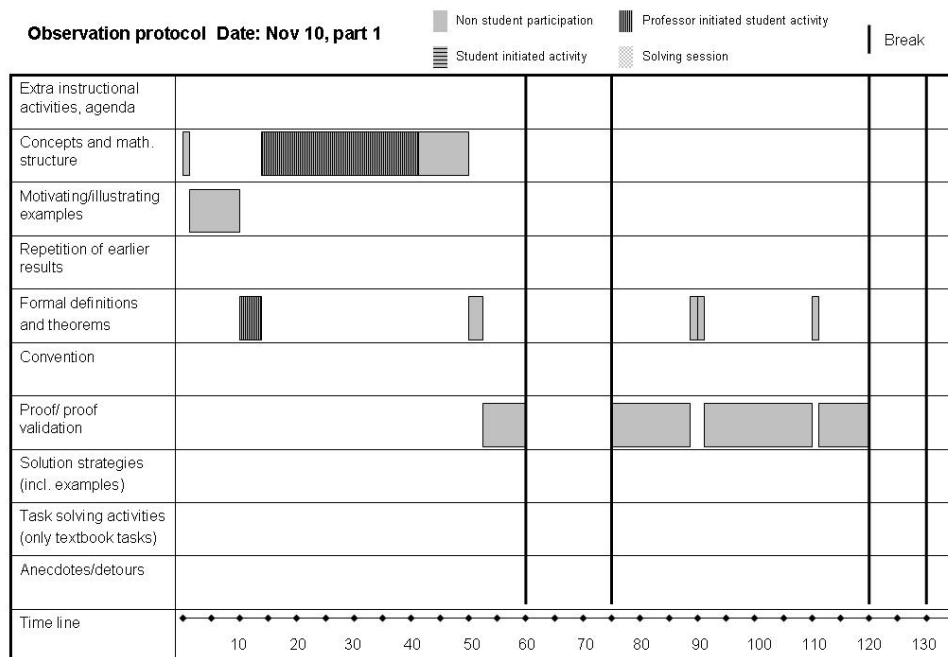
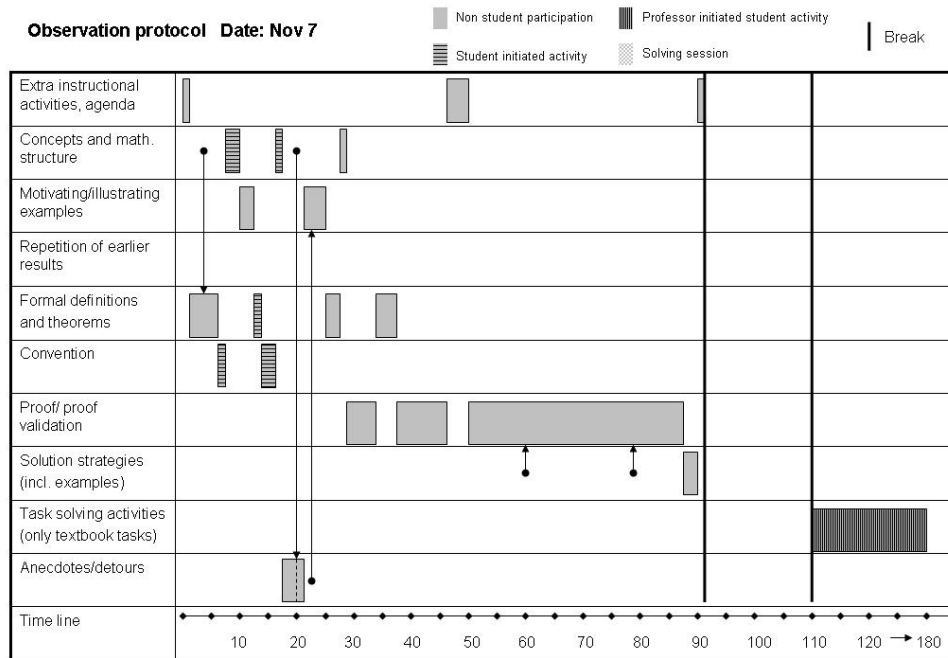




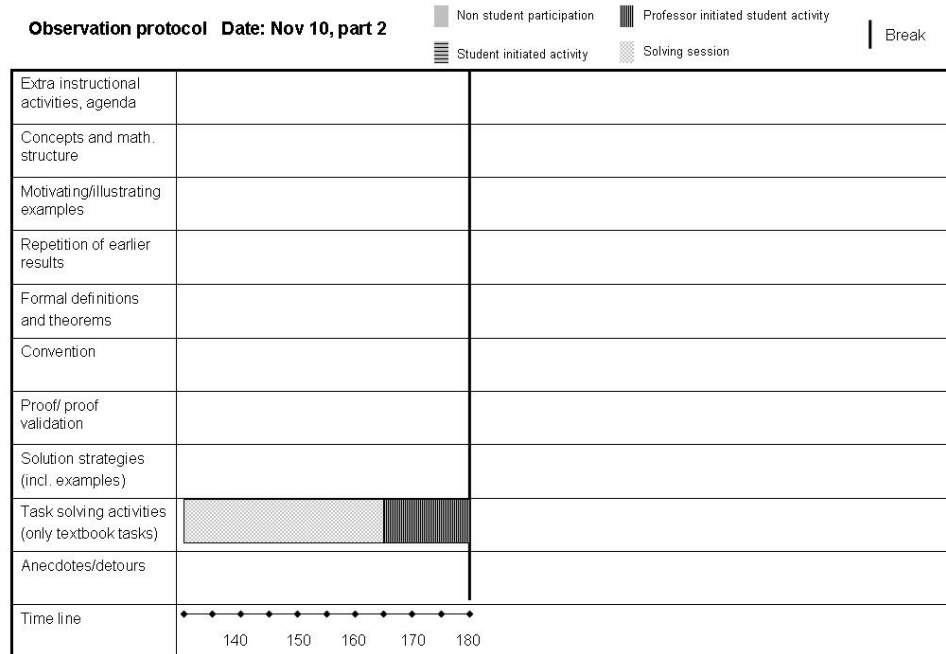
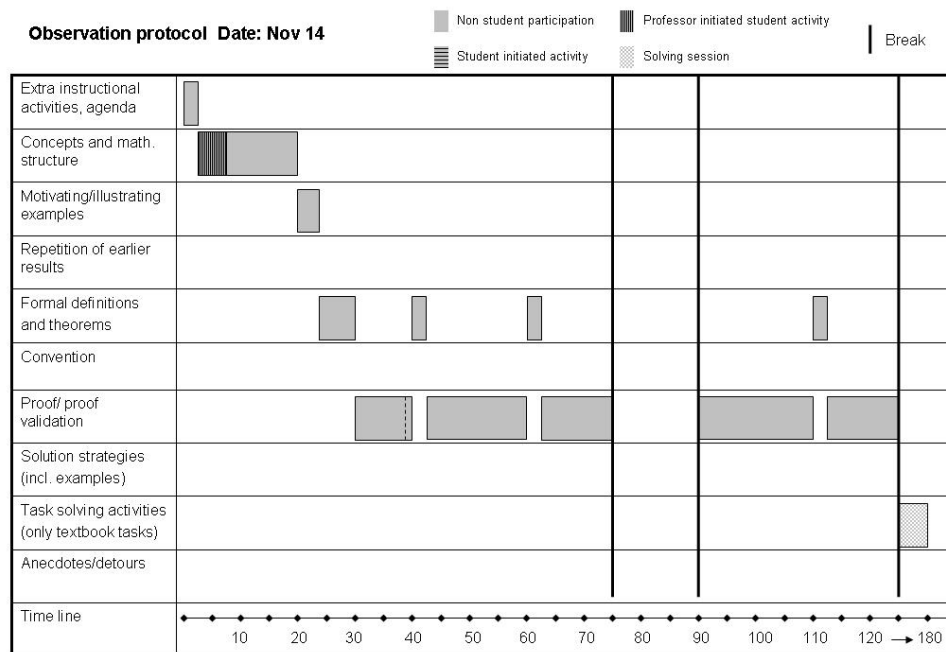


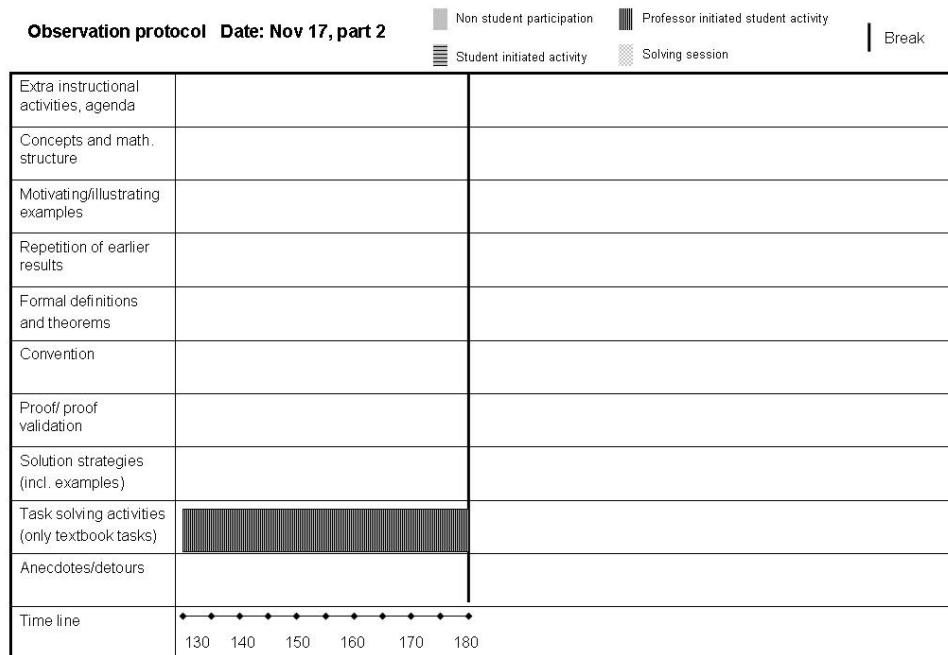
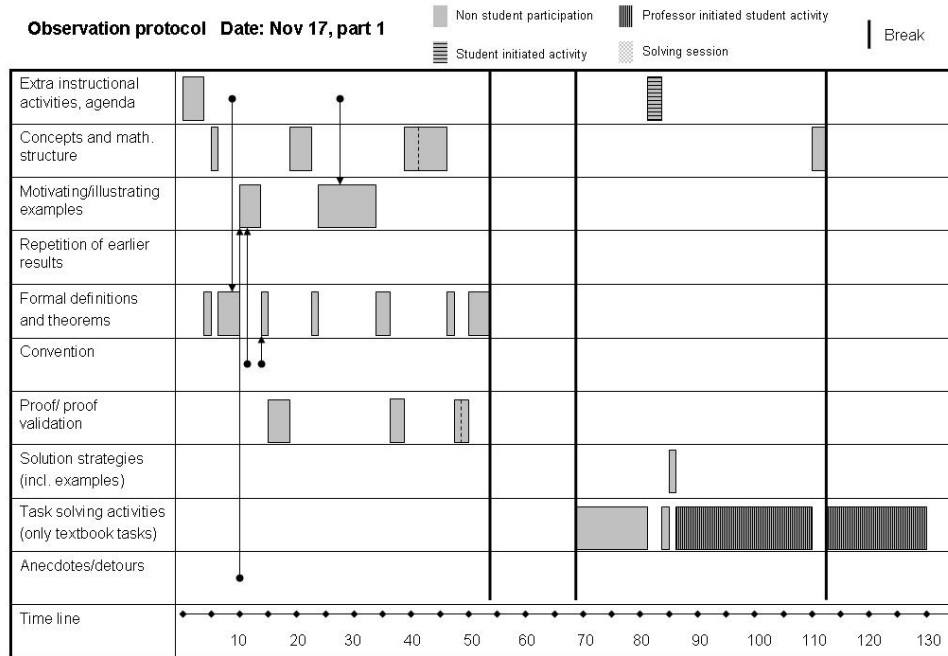


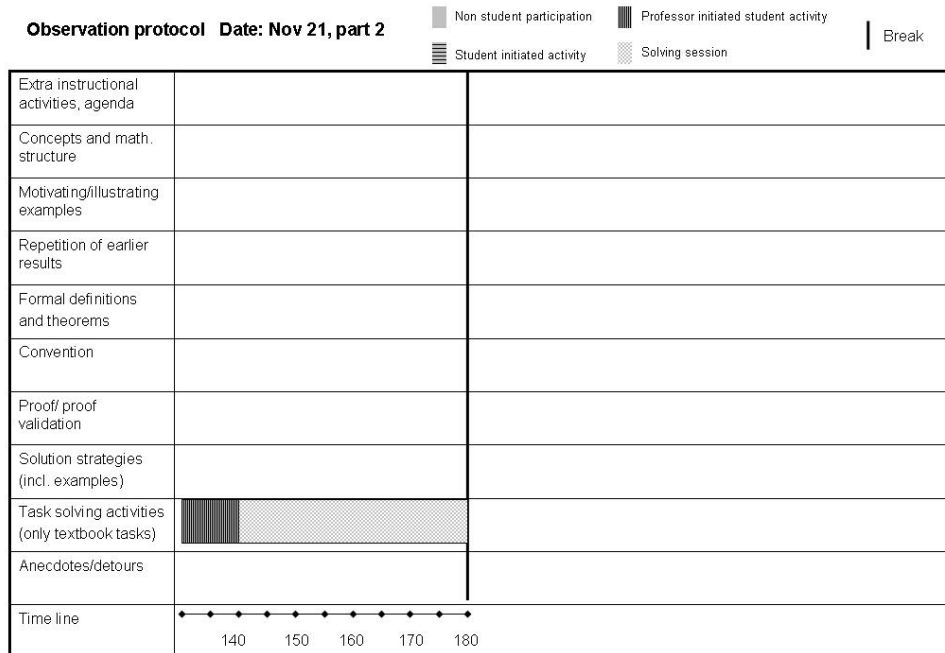
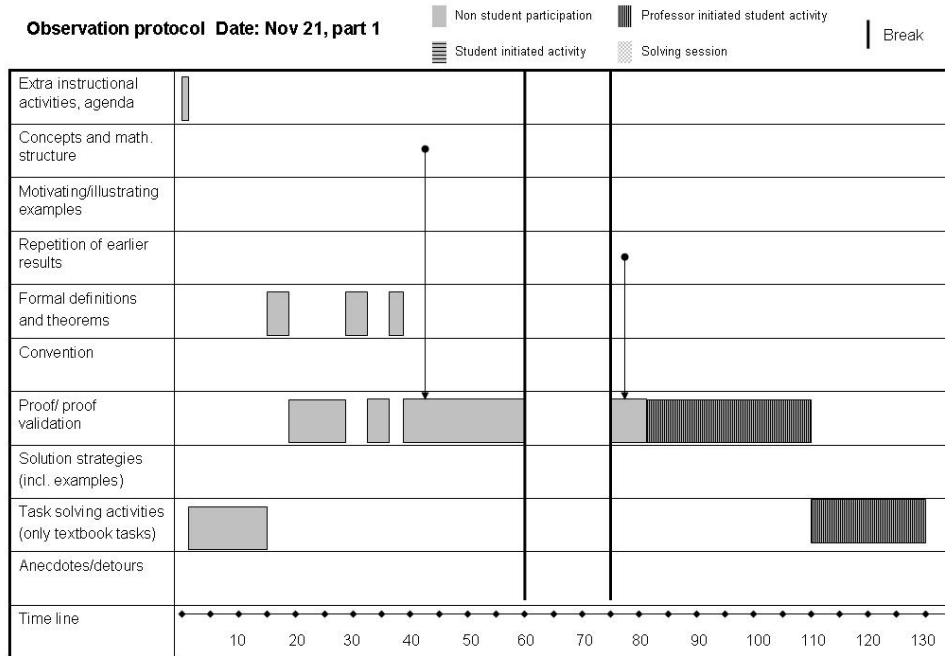


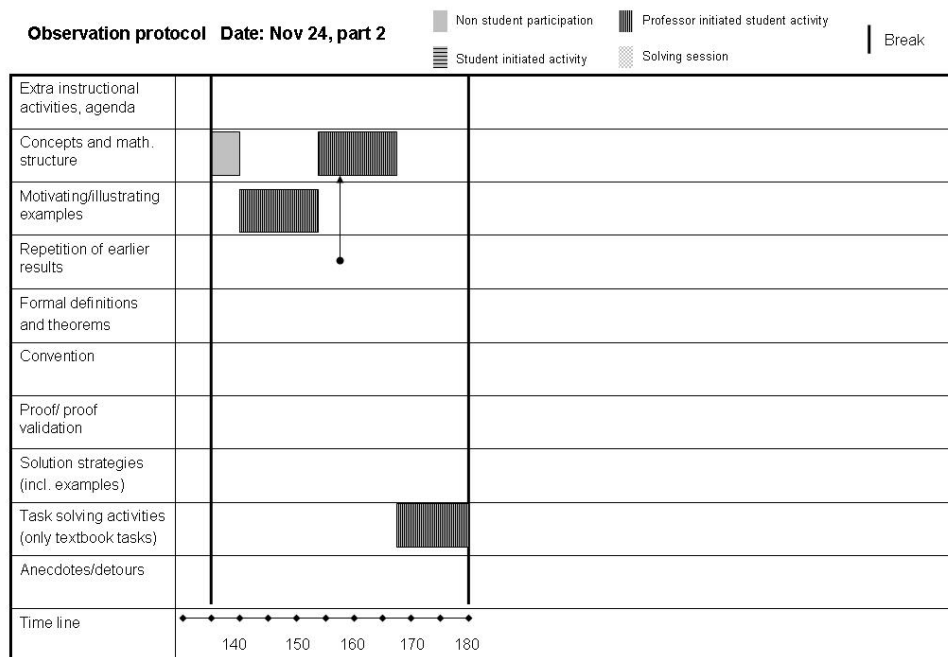


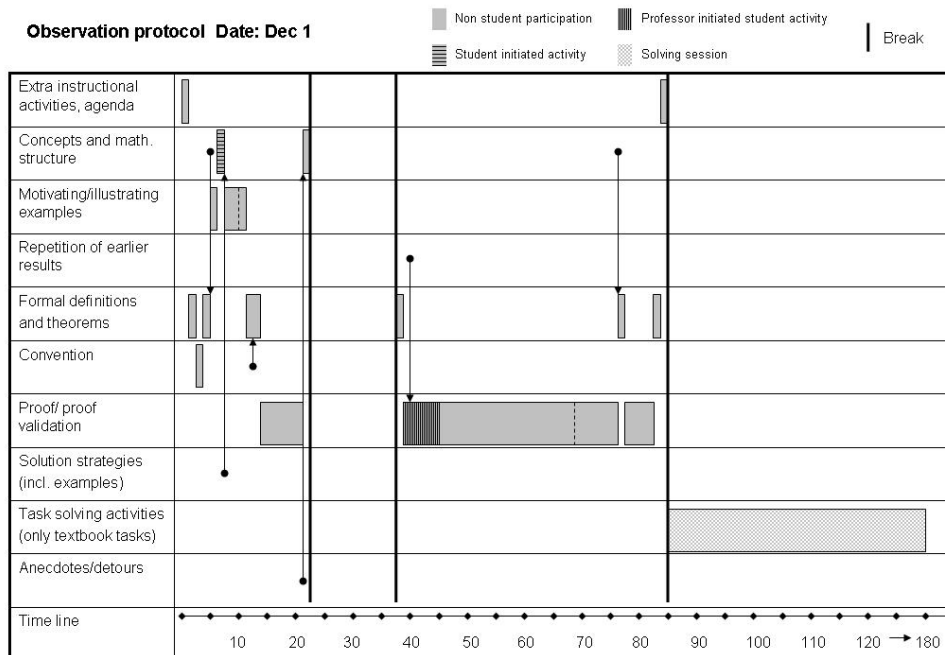
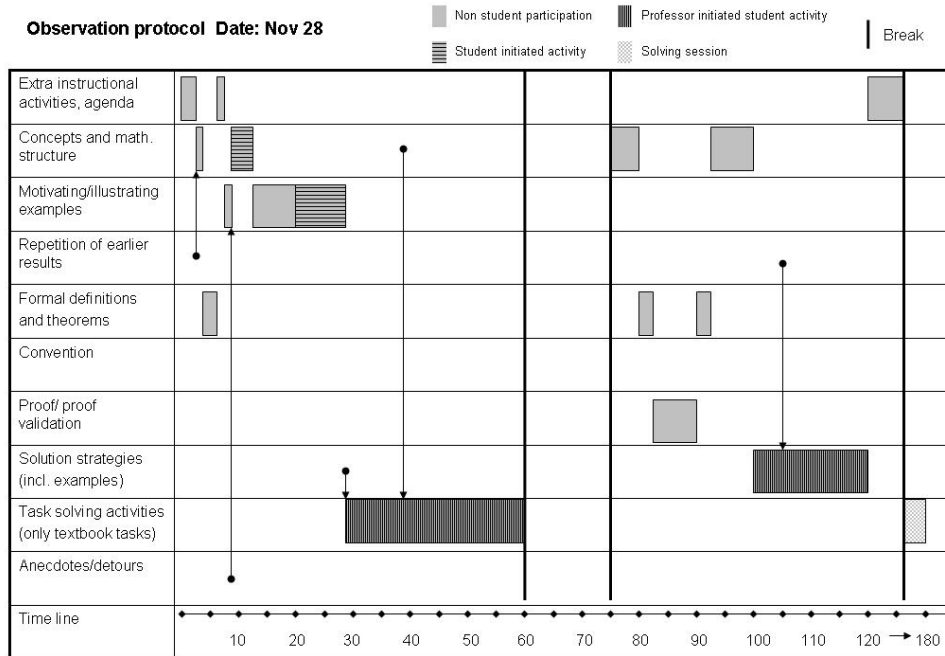


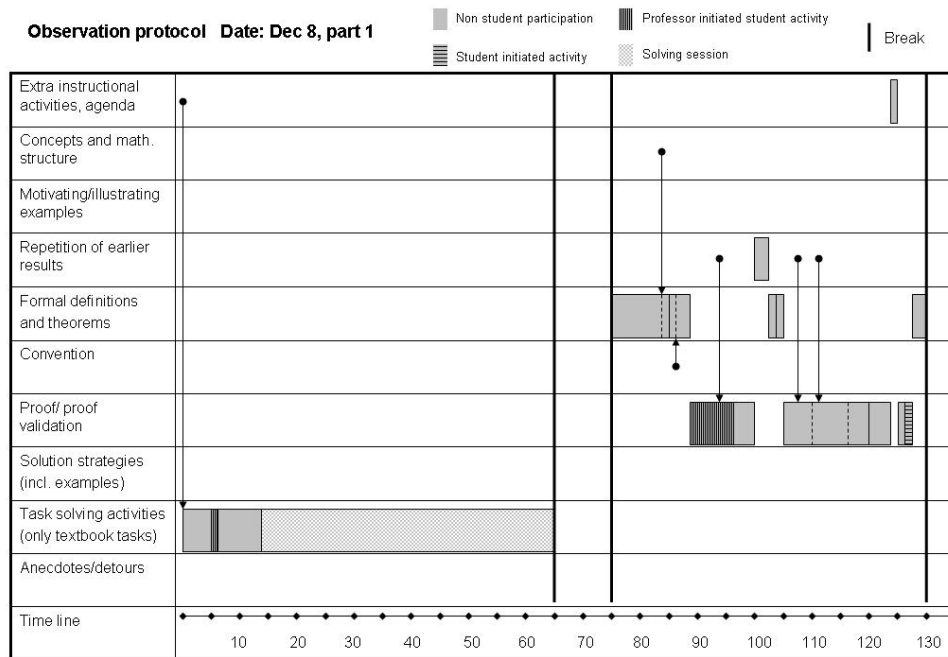
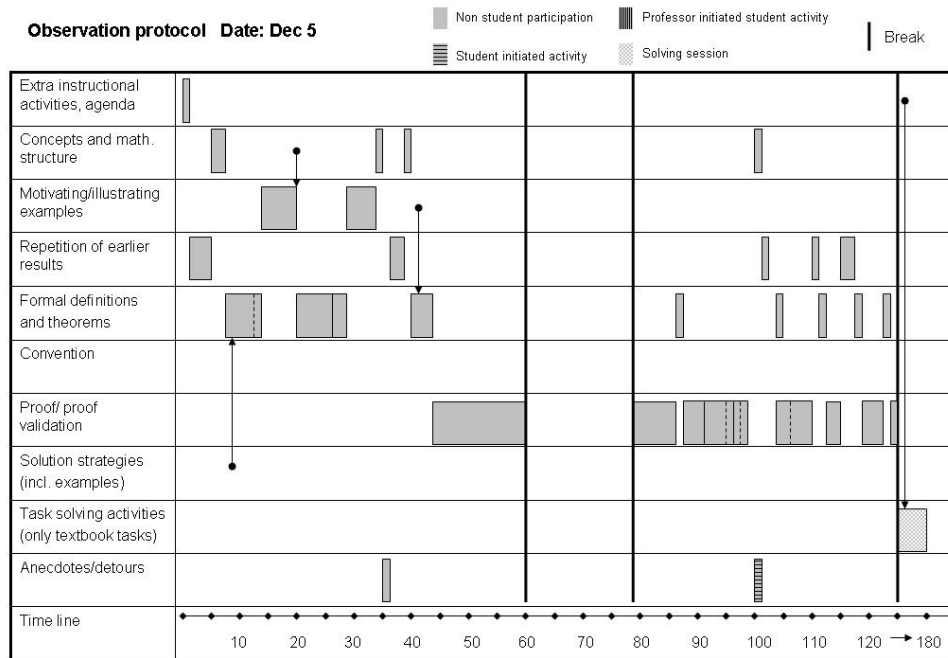
**Observation protocol Date: Nov 10, part 2****Observation protocol Date: Nov 14**

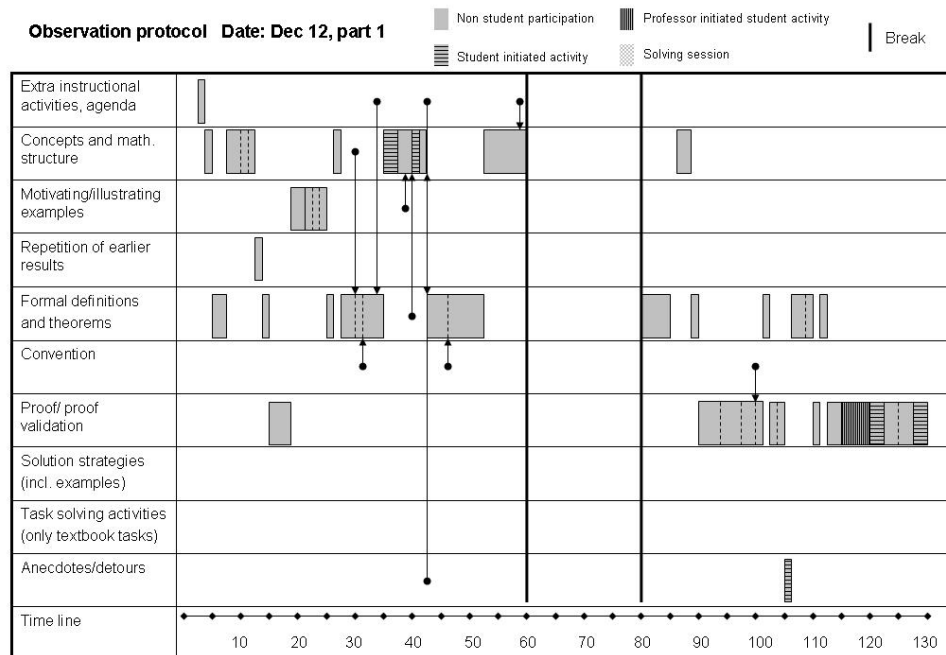
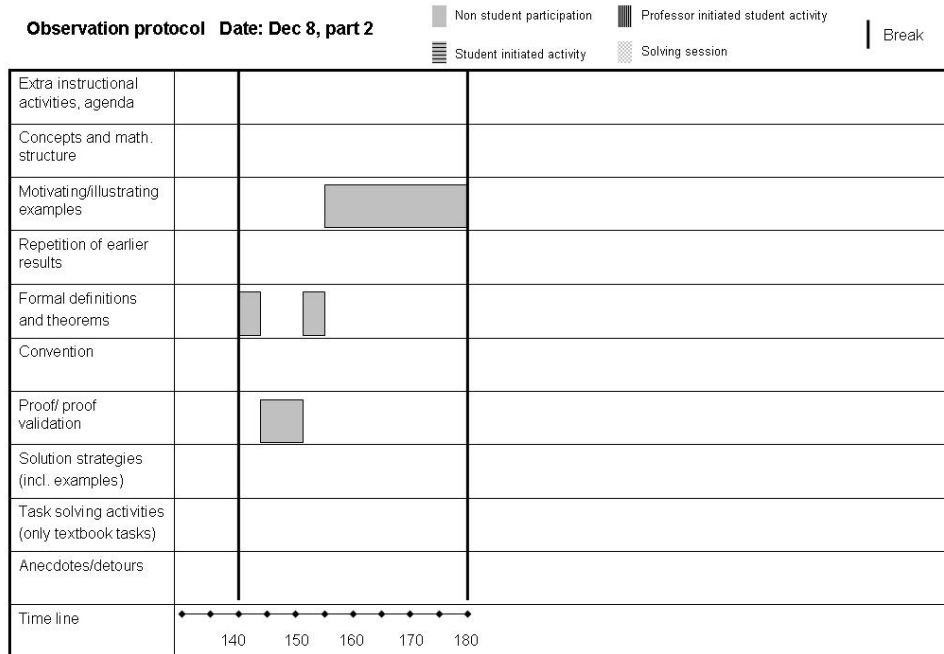


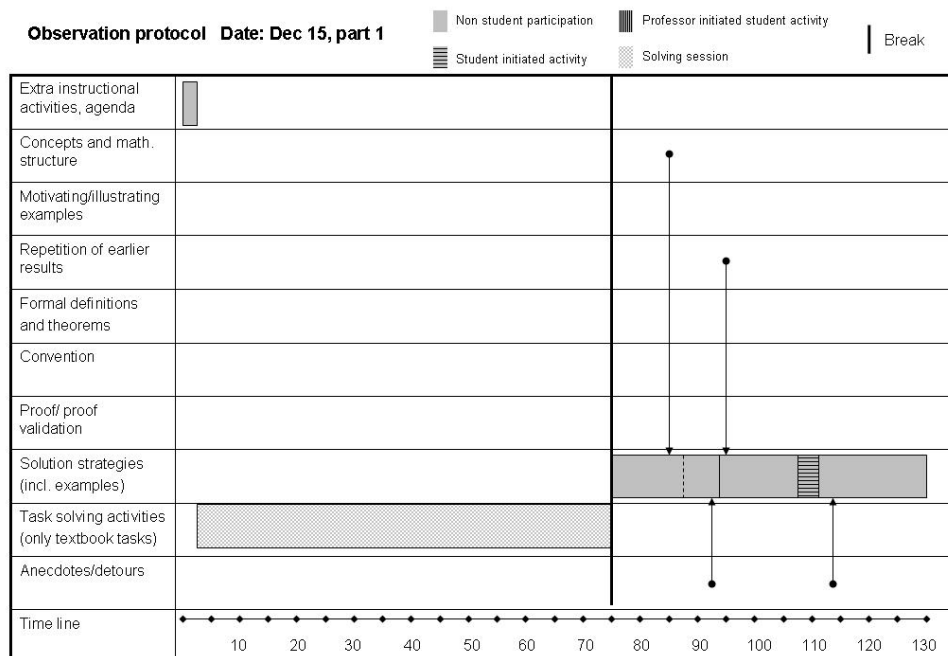
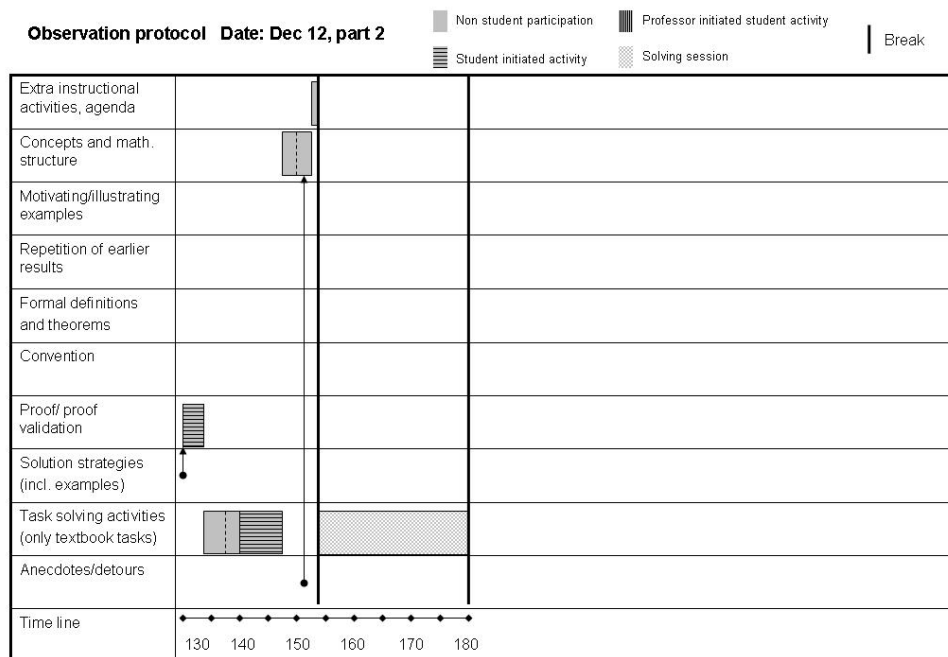



















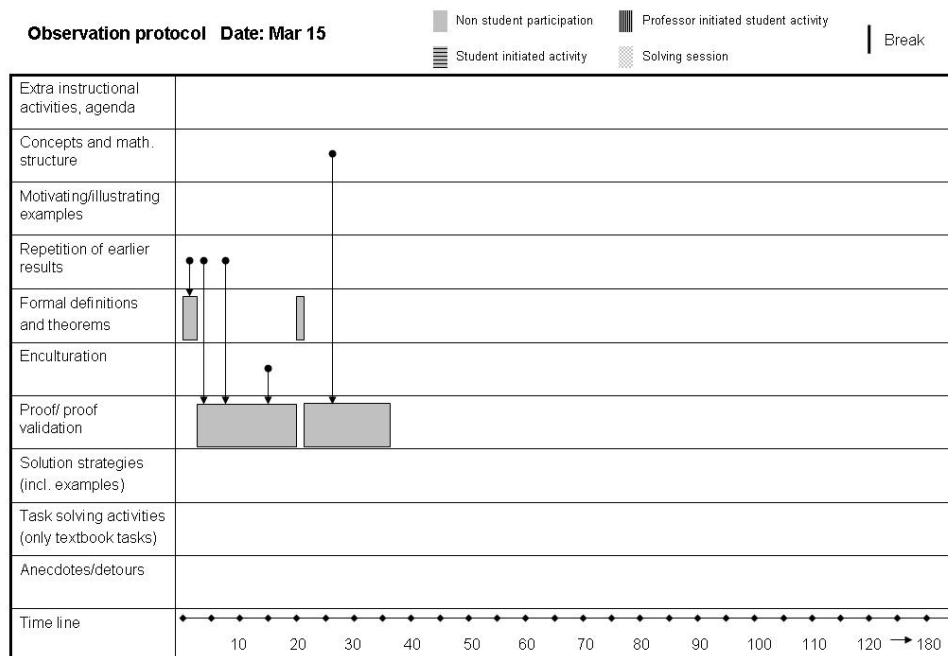
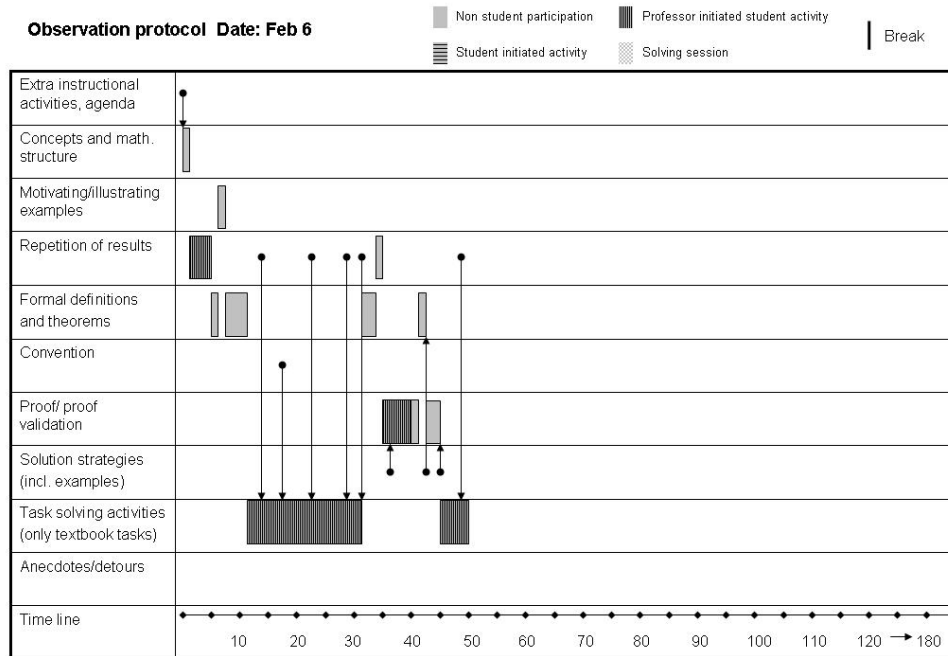


**Observation protocol** Date: Dec 15, part 2

 Non student participation   
  Professor initiated student activity   
 | Break  
 Student initiated activity   
 Solving session

Extra instructional activities, agenda			
Concepts and math. structure			
Motivating/illustrating examples			
Repetition of earlier results			
Formal definitions and theorems			
Convention			
Proof/ proof validation			
Solution strategies (incl. examples)			
Task solving activities (only textbook tasks)			
Anecdotes/detours			
Time line	 140    150    160    170    180		

## B.2 Supplementary study





# C Relevant definitions and theorems

This appendix contains the definitions and theorems mentioned in the main text. Unless otherwise stated, they are taken from the textbook [Wade, 2004].

## C.1 Definitions

**Definition of the outer Lebesgue measure** The outer Lebesgue measure on  $\mathbb{R}^n$ ,  $m^* : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty]$  is given by  $m^*(\emptyset) = 0$  and more generally

$$\forall E \subseteq \mathbb{R}^n : \quad m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} V(I_k) \mid I_k = I_{a_k, b_k}, E \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

(translated from the professor's note 4, pilot study)

**2.1 Definition** A sequence of real numbers  $\{x_n\}$  is said to *converge* to a real number  $a \in \mathbb{R}$  if and only if for every  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  (which in general depends on  $\epsilon$ ) such that

$$n \leq N \quad \text{implies} \quad |x_n - a| < \epsilon.$$

**3.1 Definition** Let  $a \in \mathbb{R}$ , let  $I$  be an open interval that contains  $a$ , and let  $f$  be a real function defined everywhere on  $I$  except possibly at  $a$ . Then  $f(x)$  is said to converge to  $L$ , as  $x$  approaches  $a$ , if and only if for every  $\epsilon > 0$  there is a  $\delta > 0$  (which in general depends on  $\epsilon$ ,  $f$ ,  $I$  and  $a$ ) such that

$$(1) \quad 0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

**3.19 Definition** Let  $E$  be a nonempty subset of  $\mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ .

- i)  $f$  is said to be *continuous at a point*  $a \in E$  if and only if given  $\epsilon > 0$  there is a  $\delta > 0$  (which in general depends on  $\epsilon$ ,  $f$ , and  $a$ ) such that

$$|x - a| < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad |f(x) - f(a)| < \epsilon. \quad (\text{C.1})$$

- ii)  $f$  is said to be *continuous on*  $E$  (notation:  $f : E \rightarrow Y$  is continuous) if and only if  $f$  is continuous at every  $x \in E$ .

**3.25 Definition** Let  $E$  be a nonempty subset of  $\mathbb{R}$ . A function  $f : E \rightarrow \mathbb{R}$  is said to be *bounded* on  $E$  if and only if there is an  $M \in \mathbb{R}$  such that  $|f(x)| \leq M$  for all  $x \in E$ .

**3.35 Definition** Let  $E$  be a nonempty subset of  $\mathbb{R}$ , and  $f : E \rightarrow \mathbb{R}$ . Then  $f$  is said to be *uniformly continuous* on  $E$  (notation:  $f : E \rightarrow \mathbb{R}$  is uniformly continuous) if and only if given  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad |f(x) - f(a)| < \epsilon.$$

**7.7 Definition** Let  $E$  be a nonempty subset of  $\mathbb{R}$ . A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  is said to *converge uniformly* on  $E$  to a function  $f$  if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

for all  $x \in E$ .

**7.13 Definition** Let  $f_k$  be a sequence of real functions defined on some set  $E$  and set

$$s_n(x) := \sum_{k=1}^n f_k(x), \quad x \in E, n \in \mathbb{N}.$$

- (i) The series  $\sum_{k=1}^{\infty} f_k$  is said to *converge pointwise* on  $E$  if and only if the sequence  $s_n(x)$  converges pointwise on  $E$  as  $n \rightarrow \infty$ .
- (ii) The series  $\sum_{k=1}^{\infty} f_k$  is said to *converge uniformly* on  $E$  if and only if the sequence  $s_n(x)$  converges uniformly on  $E$  as  $n \rightarrow \infty$ .
- (iii) The series  $\sum_{k=1}^{\infty} f_k$  is said to *converge absolutely (pointwise)* on  $E$  if and only if  $\sum_{k=1}^{\infty} |f_k(x)|$  converges for each  $x \in E$ .

**10.27 Definition** Let  $E$  be a nonempty subset of  $X$  and  $f : E \rightarrow Y$ .

- i)  $f$  is said to be *continuous at a point*  $a \in E$  if and only if given  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$\rho(x, a) < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad \tau(f(x), f(a)) < \epsilon. \quad (\text{C.2})$$

- ii)  $f$  is said to be *continuous on*  $E$  (notation:  $f : E \rightarrow Y$  is continuous) if and only if  $f$  is continuous at every  $x \in E$ .

**10.51 Definition** Let  $X$  be a metric space,  $E$  be a nonempty subset of  $X$ , and  $f : E \rightarrow Y$ . Then  $f$  is said to be *uniformly continuous* on  $E$  (notation:  $f : E \rightarrow Y$  is uniformly continuous) if and only if given  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$\rho(x, a) < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad \tau(f(x), f(a)) < \epsilon.$$

**13.1 Definition** Let  $x \in \mathbb{R}$  and let  $\epsilon > 0$ . A *neighborhood* of  $x$  is a set of the form

$$N_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\}.$$

[Lay, 1990, p. 105]

**13.2 Definition** Let  $x \in \mathbb{R}$  and let  $\epsilon > 0$ . A *deleted neighborhood* of  $x$  is a set of the form

$$N_\epsilon^*(x) = \{y \in \mathbb{R} : 0 < |x - y| < \epsilon\}.$$

[Lay, 1990, p. 106]

**13.3 Definition** Let  $S$  be a subset of  $\mathbb{R}$ . A point  $x \in \mathbb{R}$  is an *interior* point of  $S$  if there exists a neighborhood  $N$  of  $x$  such that  $N \subseteq S$ . If for every neighborhood  $N$  of  $x$ ,  $N \cap S \neq \emptyset$  and  $N \cap \mathbb{R} \setminus S \neq \emptyset$ , then  $x$  is called a *boundary* point of  $S$ . The set of all interior points of  $S$  is denoted by  $\text{int } S$ , and the set of all boundary points of  $S$  is denoted by  $\text{bd } S$ . [Lay, 1990, p. 106]

**13.6 Definition** Let  $S \subseteq \mathbb{R}$ . If  $\text{bd } S \subseteq S$ , then  $S$  is said to be *closed*. If  $\text{bd } S \subseteq \mathbb{R} \setminus S$ , then  $S$  is said to be *open*. [Lay, 1990, p. 107]

## C.2 Theorems

**2.8 Theorem** Every convergent sequence is bounded.

**2.9 Theorem** [Squeeze theorem] Suppose that  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{w_n\}$  are real sequences,

- (1) If  $x_n \rightarrow a$  and  $y_n \rightarrow a$  (the SAME  $a$ ) as  $n \rightarrow \infty$ , and if there is an  $N_0 \in \mathbb{N}$  such that

$$x_n \leq w_n \leq y_n \quad \text{for } n \geq N_0,$$

then  $w_n \rightarrow a$  as  $n \rightarrow \infty$ .

- (2) If  $x_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\{y_n\}$  is bounded, then  $x_n y_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**3.4 Remark** Let  $a \in \mathbb{R}$ , let  $I$  be an open interval that contains  $a$ , and let  $f, g$  be real functions defined everywhere on  $I$  except possibly at  $a$ . If  $f(x) = g(x)$  for all  $x \in I \setminus \{a\}$  and  $f(x) \rightarrow L$  as  $x \rightarrow a$ , then  $g(x)$  also has a limit as  $x \rightarrow a$  and

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x).$$

**3.6 Theorem** [Sequential Characterisation of Limits] Let  $a \in \mathbb{R}$ , let  $I$  be an open interval that contains  $a$ , and let  $f$  be a real function defined everywhere on  $I$  except possibly at  $a$ . Then

$$L = \lim_{x \rightarrow a} f(x)$$

exists if and only if  $f(x_n) \rightarrow L$  as  $n \rightarrow \infty$  for every sequence  $x_n \in I \setminus \{a\}$  that converges to  $a$  as  $n \rightarrow \infty$ .

**3.26 Theorem** [Extreme Value Theorem] If  $I$  is a closed, bounded interval and  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$ , then  $f$  is bounded on  $I$ . Moreover, if

$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x),$$

then there exist points  $x_m, x_M \in I$  such that

$$f(x_M) = M \quad \text{and} \quad f(x_m) = m.$$

**3.27 Remark** The Extreme Value Theorem is false if either ‘closed’ or ‘bounded’ is dropped from the hypotheses.

**3.39 Theorem** Suppose that  $I$  is a closed, bounded interval. If  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$ , then  $f$  is uniformly continuous on  $I$ .

**5.10 Theorem** Suppose that  $a, b \in \mathbb{R}$  with  $a < b$ . If  $f$  is continuous on the interval  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .

**5.28 Theorem** [Fundamental Theorem of Calculus] Let  $[a, b]$  be nondegenerate and suppose that  $f : [a, b] \rightarrow \mathbb{R}$ .

- (i) If  $f$  is continuous on  $[a, b]$  and  $F(x) = \int_a^x f(t) dt$ , then  $F \in \mathcal{C}^1[a, b]$  and

$$\frac{d}{dx} \int_a^x f(t) dt := F'(x) = f(x)$$

for each  $x \in [a, b]$ .

(ii) If  $f$  is differentiable on  $[a, b]$  and  $f'$  is integrable on  $[a, b]$ , then

$$\int_a^x f'(t) dt = f(x) - f(a)$$

for each  $x \in [a, b]$ .

**5.34 Theorem** [Change of Variables] Let  $\phi$  be continuously differentiable on a closed, nondegenerate interval  $[a, b]$ . If

$f$  is continuous on  $\phi([a, b])$ ,

or if

$\phi$  is strictly increasing on  $[a, b]$  and  $f$  is integrable on  $[\phi(a), \phi(b)]$ ,

then

$$\int_{\phi(a)}^{\phi(b)} f(t) dt = \int_a^b f(\phi(x)) \phi'(x) dx.$$

**7.9 Theorem** Let  $E$  be a nonempty subset of  $\mathbb{R}$  and suppose that  $f_n \rightarrow f$  uniformly on  $E$ . If each  $f_n$  is continuous at some  $x_0 \in E$ , then  $f$  is continuous at  $x_0 \in E$ .

**7.10 Theorem** Suppose that  $f_n \rightarrow f$  uniformly on a closed interval  $[a, b]$ . If each  $f_n$  is integrable on  $[a, b]$ , then so is  $f$  and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \left( \lim_{n \rightarrow \infty} f_n(x) \right) dx.$$

In fact,  $\lim_{n \rightarrow \infty} \int_a^x f_n(t) dt = \int_a^x f(t) dt$  for  $x \in [a, b]$ .

**7.11 Theorem** [Uniform Cauchy Criterion] Let  $E$  be a nonempty subset of  $\mathbb{R}$  and let  $f_n : E \rightarrow \mathbb{R}$  be a sequence of functions. Then  $f_n$  converges uniformly on  $E$  if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$n, m \geq N \quad \text{imply} \quad |f_n(x) - f_m(x)| < \epsilon$$

for all  $x \in E$ .

**7.12 Theorem** Let  $(a, b)$  be a bounded interval and suppose that  $f_n$  is a sequence of functions that converges at some  $x_0 \in (a, b)$ . If each  $f_n$  is differentiable on  $(a, b)$ , and  $f'_n$  converges uniformly on  $(a, b)$  as  $n \rightarrow \infty$ , then  $f_n$  converges uniformly on  $(a, b)$  and

$$\lim_{n \rightarrow \infty} f'_n(x) = \left( \lim_{n \rightarrow \infty} f_n(x) \right)'$$

for each  $x \in (a, b)$ .

**7.14 Theorem** Let  $E$  be a nonempty subset of  $\mathbb{R}$ , and let  $\{f_k\}$  be a sequence of real functions defined on  $E$ .

- (i) Suppose that  $x_0 \in E$  and that each  $f_k$  is continuous at  $x_0 \in E$ . If  $f = \sum_{k=1}^{\infty} f_k$  converges uniformly on  $E$ , then  $f$  is continuous at  $x_0 \in E$ .
- (ii) [Term-by-term integration]
- (iii) [Term-by-term differentiation]

**7.15 Theorem** [Weierstrass M-test] Let  $E$  be a nonempty subset of  $\mathbb{R}$ , let  $f_k : E \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$ , and let  $M_k \geq 0$  satisfy  $\sum_{k=1}^{\infty} M_k < \infty$ . If  $|f_k(x)| \leq M_k$  for  $k \in \mathbb{N}$  and  $x \in E$ , then  $\sum_{k=1}^{\infty} f_k$  converges absolutely and uniformly on  $E$ .

**7.16 Theorem** [Dirichlet's test for uniform convergence] Let  $E$  be a nonempty subset of  $\mathbb{R}$  and suppose that  $f_k, g_k : E \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$ . If

$$\left| \sum_{k=1}^{\infty} f_k(x) \right| \leq M < \infty$$

for  $n \in \mathbb{N}$  and  $x \in E$ , and if  $g_k \downarrow 0$  uniformly on  $E$  as  $k \rightarrow \infty$ , then  $\sum_{k=1}^{\infty} f_k g_k$  converges uniformly on  $E$ .

**7.45 Theorem** Let  $f \in C^\infty(a, b)$ . If there is an  $M > 0$  such that

$$|f^{(n)}(x)| \leq M^n$$

for all  $x \in (a, b)$  and  $n \in \mathbb{N}$ , then  $f$  is analytic on  $(a, b)$ . In fact, for each  $x_0 \in (a, b)$ ,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

holds for all  $x \in (a, b)$ .

**10.28 Theorem** Let  $E$  be a nonempty subset of  $X$  and  $f, g : E \rightarrow Y$ .

- (i)  $f$  is continuous at  $a \in E$  if and only if  $f(x_n) \rightarrow f(a)$ , as  $n \rightarrow \infty$ , for all sequences  $x_n \in E$  that converge to  $a$ .
- (ii) Suppose that  $Y = \mathbb{R}^n$ . If  $f, g$  are continuous at a point  $a \in E$  (respectively, continuous on a set  $E$ ), then so are  $f + g$ ,  $f \cdot g$ , and  $\alpha f$  (for any  $\alpha \in \mathbb{R}$ ). Moreover, in the case  $Y = \mathbb{R}$ ,  $f/g$  is continuous at  $a \in E$  when  $g(a) \neq 0$  (respectively, on  $E$  when  $g(x) \neq 0$  for all  $x \in E$ ).

**10.52 Theorem** Suppose that  $E$  is a compact subset of  $X$  and  $f : X \rightarrow Y$ . Then  $f$  is uniformly continuous on  $E$  if and only if  $f$  is continuous on  $E$ .

**10.58 Theorem** Let  $X$  and  $Y$  be metric spaces, and let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if  $f^{-1}(V)$  is open in  $X$  for every open  $V$  in  $Y$ .

**11.20 Theorem** Let  $\alpha \in \mathbb{R}$ ,  $\mathbf{a} \in \mathbb{R}^n$ , and suppose that  $f$  and  $g$  are vector functions. If  $f$  and  $g$  are differentiable at  $\mathbf{a}$  then  $f + g$ ,  $\alpha f$ , and  $f \cdot g$  are all differentiable at  $\mathbf{a}$ . In fact,

$$\begin{aligned} D(f + g)(\mathbf{a}) &= Df(\mathbf{a}) + Dg(\mathbf{a}), \\ D(\alpha f)(\mathbf{a}) &= \alpha Df(\mathbf{a}), \end{aligned}$$

and

$$D(f \cdot g)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}).$$

**16.1 Proposition** Let  $(I_n)$  and  $(J_k)$  be sequences of intervals such that  $\bigcup_{n=1}^{\infty} I_n = \bigcup_{k=1}^{\infty} J_k$ . If the  $I_n$  are pairwise disjoint, then  $\sum_{n=1}^{\infty} l(I_n) \leq \sum_{k=1}^{\infty} l(J_k)$ . Thus, if the  $J_k$  are also pairwise disjoint, the two sums are equal. (from [Carothers, 2000, p. 268])

**16.4 Proposition**  $m^*(I) = \ell(I)$  for any interval  $I$ , bounded or not. (from [Carothers, 2000, p. 270])





## D Tasks for investigating solving processes

### D.1 Task in the preliminary study

Determine all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , which fulfil  $f(x) - f(y) \leq (x - y)^2$  for all  $x, y \in \mathbb{R}$ .

### D.2 Tasks in the pilot study

#### Task 1

Let  $(M, \sigma_{discrete})$  and  $(M, \sigma_d)$  be two metric spaces, where  $\sigma_{discrete}$  and  $\sigma_d$  are the discrete metric and an arbitrary metric, respectively. Let  $i$  be the identity, i.e.  $i(x) = x$ ,  $x \in M$ .

Determine if the mapping  $i : (M, \sigma_{discrete}) \rightarrow (M, \sigma_d)$  is continuous and uniformly continuous.

#### Task 2

Assume that there for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  exists a constant  $K$ , such that  $|f(x) - f(y)| \leq K|x - y|$ , for all  $x, y \in \mathbb{R}$ . Show that  $m^*(f(E)) \leq Km^*(E)$ , for all  $E \subseteq \mathbb{R}$ , where  $m^*$  is the outer Lebesgue measure.

#### Task 3

Consider the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ :

$$f(x, y) = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{else} \end{cases}$$

1. At which points in  $\mathbb{R}^2$  does  $f$  have partial derivatives?
2. At which points in  $\mathbb{R}^2$  is  $f$  continuous?
3. At which points in  $\mathbb{R}^2$  is  $f$  differentiable?
4. At which points in  $\mathbb{R}^2$  does  $f$  have directional derivatives in any direction?

Answer the same questions for the function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ :

$$g(x, y) = \begin{cases} x^2 & \text{if } y = 0 \\ y^2 & \text{if } x = 0 \\ 0 & \text{else} \end{cases}$$

#### Task 4

Is the set of intervals of  $\mathbb{R}$  (i.e. sets of the form  $[a, b]$ ,  $]a, b[$ ,  $[a, \infty[$ ,  $]a, \infty[$ ,  $] - \infty, a]$ ,  $] - \infty, a[$ ,  $] - \infty, \infty[$ ,  $a, b \in \mathbb{R}$ ) a  $\sigma$ -algebra?

### D.3 Task in the main study

A sequence of functions  $\{f_n\}$  is said to be *uniformly bounded* on an interval  $[a, b]$  if and only if there exists a number  $M > 0$  such that

$$|f_n(x)| \leq M$$

for all  $n$  and for all  $x \in [a, b]$ .

- Show that a uniformly convergent sequence  $\{f_n\}$  of continuous functions on  $[a, b]$  is uniformly bounded.
  - Show that the statement is true only if the interval is closed and bounded.
  - Show that the statement is true only if the sequence is uniformly convergent.

### D.4 Tasks in the supplementary study

#### Task 1

Let  $f$  be a differentiable and even function. Proof whether  $f'$  is either even, odd, or neither of the two.

##### Definition of even and odd functions

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called **even** if  $f(-x) = f(x)$  for all  $x \in \mathbb{R}$ .

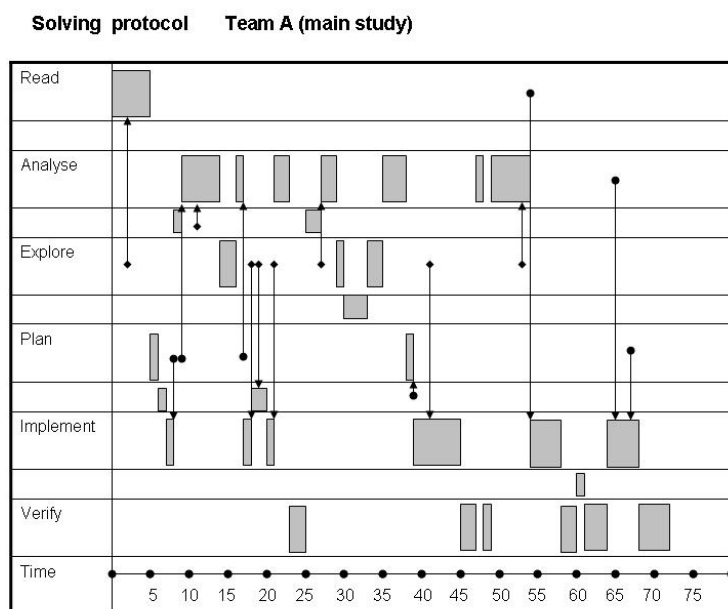
A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called **odd** if  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$ .

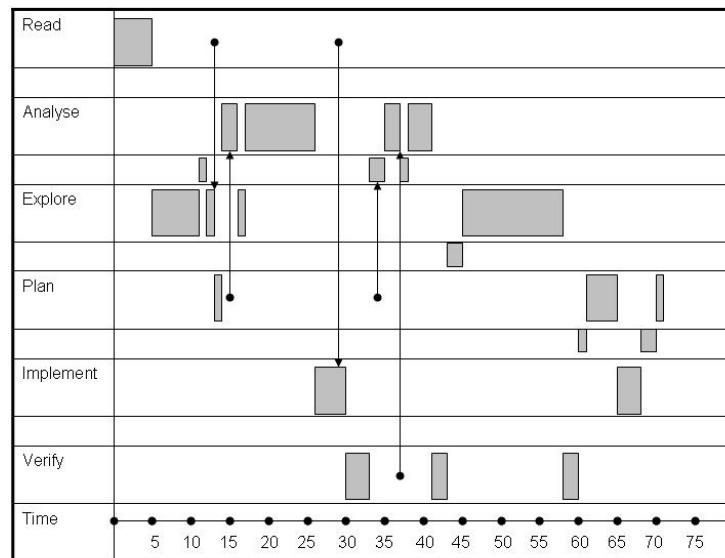
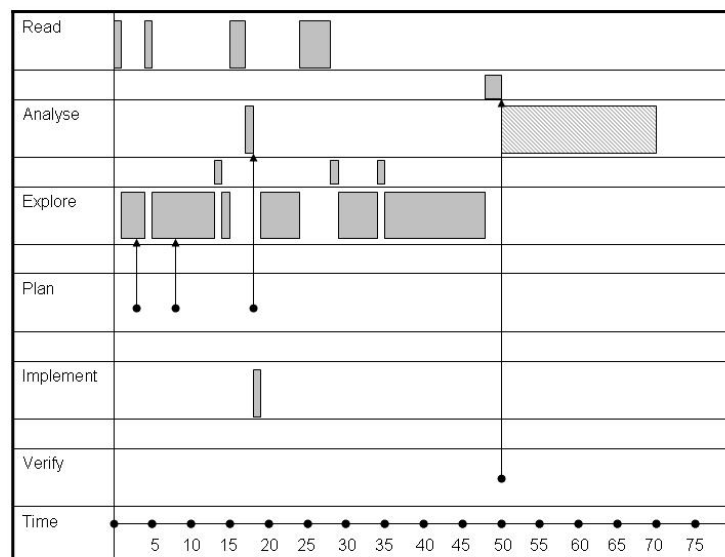
#### Task 2

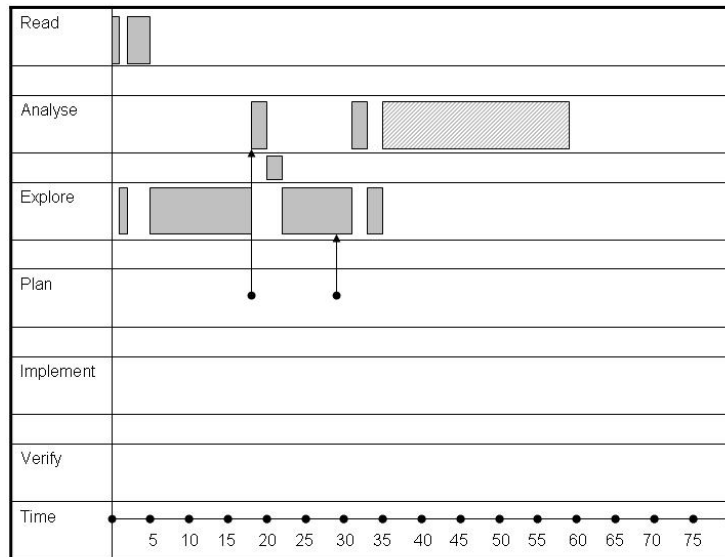
Determine all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , which fulfil  $f(x) - f(y) \leq (x - y)^2$  for all  $x, y \in \mathbb{R}$ .

## E Time-line representations of solving protocols

This appendix contains time-line representations for the four solving processes in the main study. In team C the students thought they managed to construct a proof, but was in fact far from reaching a complete proof. In team D, the students gave up constructing a proof. I tried to guide both teams C and D toward a solution, which is indicated by the change of pattern of the last analysis period in those two time-line representations.



**Solving protocol Team B (main study)****Solving protocol Team C (main study)**

**Solving protocol    Team D (main study)**



## F Student's notes, main study

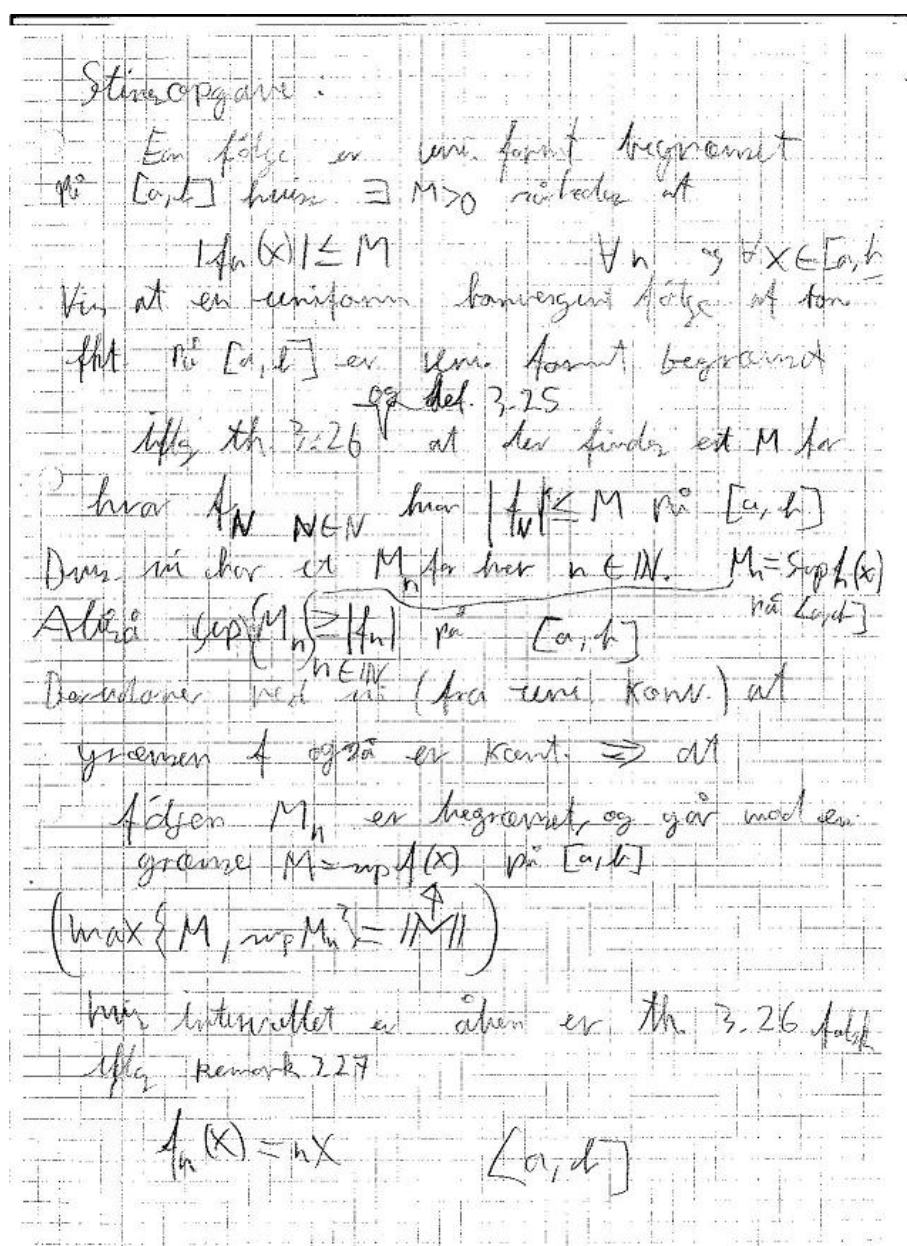


Figure F.1 Adam's notes.



①  $\{f_n\}$  er en uniform konvergent følge

Th 3.7  $f_n \xrightarrow{n \rightarrow \infty} f$  hvis

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

Th. 3.39  $I$  er et lukket, begrænset interval.  
 $f: I \rightarrow \mathbb{R}$  er kont på  $I$ , da er  $f$  uniform kont på  $I$ .

$f_n$  er en følge af kont. funktioner ~~på  $[a, b]$~~   
 $\Rightarrow$  jf Th. 3.39.  $f_n$  er en følge af uniform kont. funktioner

Th 3.26  $\Rightarrow f_n$  er begrænset på  $I$ , fast  $n$ ?

Når  $n \rightarrow \infty$ :  $f_n \Rightarrow f$

②  $f_n$  uniform kont på  $I$

$$\forall \varepsilon > 0 \exists \delta > 0: |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Brug ① og ② til at vise at  $f$  er begrænset

① Vælg  $N$ , så stor at for  $n \geq N$ :  $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$

② Vælg  $\delta$  så lille, at for

Figure F.2 Aaron's notes, page 1.

Th. 3.26: Hvis  $I$  er et lukket, begrænset interval  
 og  $f: I \rightarrow \mathbb{R}$  er kont på  $I$ , da er  $f$  begrænset  
 på  $I$   
 $\Rightarrow$  for fast men vilk.  $n \in \mathbb{N}$ ,  $\forall x \in [a, b]$ :  $f_n$  er kont på  $[a, b] \Rightarrow$   
 $f_n$  begrænset på  $[a, b]$   
 Hvad sker for  $n \rightarrow \infty$   
 def. Th. 7.9: Da  $f_n \Rightarrow f$  og  $f_n$  kont  $\forall x \in [a, b]$   
 $\Rightarrow f$  er kont  $\forall x \in [a, b] \Rightarrow$   
 def. Th. 3.26 at  $f$  er begrænset.  
 $\therefore |f_n| < M$

---

Figure F.3 Aaron's notes, page 2.

## Opgave

En følge af funktioner  $\{f_n\}$  siges at være *uniformt begrænset* på et interval  $[a, b]$  hvis der findes et tal  $M > 0$  således at

$$|f_n(x)| \leq M$$

for alle  $n$  og for alle  $x \in [a, b]$ .

- Vis at en uniform konvergent følge  $\{f_n\}$  af kontinuerlige funktioner på  $[a, b]$  er uniformt begrænset.

- Vis at udsagnet kun er sandt, hvis intervallet er afsluttet og begrænset.
- Vis at udsagnet kun er sandt, hvis følgen er uniform konvergent.

$$\limsup_{n \rightarrow \infty} f_n(x) \leq M$$

$$|f_n(x) - M| \leq C$$

$$|f_n(x) - M| \leq |f_n(x) - f(x)| + |f(x) - M| < \varepsilon$$

just et  $\varepsilon$  vælger et  $N \in \mathbb{N}$

$$|f_n(x) - f(x)| \leq |f_n(x)| - |f(x)| < \varepsilon$$

$n \in \mathbb{N}$ :

$$\varepsilon > |f_n(x) - f(x)| \geq |f_n(x)| - |f(x)| \quad \forall x \in [a, b]$$

$$M = \varepsilon + |f(x)| \Rightarrow |f_n(x)| > 2|f(x)| - \varepsilon$$

for  $\forall x \in [a, b]$

$$2 + |f(x)|$$

$$1 + 0 > 1$$

$$1 > \frac{x}{n} = 2$$

Figure F.4 Benny's notes, page 1.

$f_n(x) \rightarrow f(x)$  for  $n \rightarrow \infty$   
 Given  $\epsilon > 0$  Vælg  $N \in \mathbb{N}$  så  $n \geq N \Rightarrow$   
 $|f_n(x) - f(x)| \leq \epsilon \Leftrightarrow |f_n(x)| \leq \epsilon + |f(x)|$   
 for  $n \geq N$   ~~$|f_n(x)| \leq \epsilon$~~   <sup>$\epsilon$  Epsilon</sup>  
 ~~$M = \epsilon + |f(x)| \Rightarrow |f_n(x)| \leq M$~~   
 Hvis  $i \leq n \leq N$  så  
 $|f_n(x)| \leq \max\{|f_1(x)|, \dots, |f_N(x)|\}$   
 Sæt  $M = \max\{|f(x)| + \epsilon, \max\{|f_1(x)|, \dots, |f_N(x)|\}\} \Rightarrow$   
 $|f_n(x)| \leq M$  Q.E.D.  
 2)

Figure F.5 Benny's notes, page 2.

$$\begin{aligned}
 & |f_n(x)| \leq M \\
 & |f_n(x) - f(x)| < \varepsilon \\
 & f(x) \Rightarrow f(x) \\
 & \text{I } [a, b] \quad f_n(x) - f(x) \xrightarrow{n \rightarrow \infty} 0 \\
 & M = \limsup |f_n(x)| \quad a \leq x \leq b \\
 & f_n(x) \leq f(x) \\
 & \lim f_n(x) \leq \lim \\
 & |f_n(x) - f(x)| \leq M \\
 & |f_n(x) - f(x)| \leq M \\
 & |f_n(x)| \leq M + |f(x)| \\
 & M = \limsup |f_n(x) - f(x)|, \quad a \leq x \leq b
 \end{aligned}$$

Figure F.6 Bob's notes.

# G Interview questions

## G.1 The pilot study

### Questions for the professor, 1st set

#### The organisation of the teaching

- Before the course began, what thoughts did you have about the organisation of the teaching?
- Is it intentionally that a lesson is always divided into a lecture part and a solving part?
- In the lectures, you go through the proofs in details. What thoughts have you had about that? Is it a consequence of the way the students prepare? Because they demand it?

#### Study behaviour

- Which preparation behaviour did you expect of the students?
- Which preparation behaviour do you think they have?
- Have the actual preparation behaviour made you adjust your original plan?
- How do you view the interplay between students' activity in and outside class?
- Most of the students read the chapter cursorily before class and only a few students look at the tasks. Some students read the chapter after you have talked about it. Is that a reasonable way to prepare?
- Some students said that they do not read the proofs because you go through them so thoroughly. Do you think that there is a difference in student learning if they try to understand the proofs themselves instead of relying on you to present them?
- Is it your impression that the weaker students are setting the agenda for the instruction?

#### Task solving

- What is the purpose of solving task?
- Why do you think the students experience difficulties with task solving?
- Why do you feel that they solve too few tasks?
- Are you going to let it influence the exam tasks, that the students have difficulties solving the tasks?
- In the first part of the course, only a few task solutions were presented at the board. Was that intentionally?
- Why do you put almost all the tasks from each chapter on the week notes?
- How many students completed the hand-in assignments? Is it the same students each time?
- What is your primary reason for choosing one task above another as hand-in assignment?

#### Change from textbook to notes

- How did you experience the shift from the textbook to the notes?

**Curriculum and competencies**

- How did you choose the course curriculum?
- Why did you choose to include measure and integration theory?
- Which competencies do you imagine that the students especially train during the course?

**Questions for the professor, 2nd set****The teaching**

- At the evaluation you asked for more student activity in the lectures. More dialogue. What did you mean by that?
- Do you think it is possible to have a ‘scientific debate’ given the way the students prepare?

**Questions for students, 1st set****Study habits**

- Do you read the chapter before class?
- Do you attempt to solve some of the assigned tasks? If no, why not?
- How many tasks do you manage to solve from each chapter?
- Do you stay to solve tasks in the solving sessions? Do you try to solve tasks at home?

**Task solving**

- What do you gain from solving tasks?
- Do you feel that the lectures are on one level and your basis for solving the tasks is at another?
- Are you satisfied with the organisation of the course?
- Is it okay that you do not have any review of solutions to assigned task in class?

**Professor’s initiative to talk about task solving**

- Do you agree with the professor that you do not manage to solve enough tasks?
- Does that give rise to frustrations?
- How do you interpret the announcement that you should solve at least four tasks from each chapter?

**Self opinion**

- How would you describe your participation in and profit of the teaching?
- If you feel that you are not quite able to understand what is going on in the classroom, what do you think the reason is?

**Transition from the previous analysis course**

- Do you see this course as a direct continuation of the previous analysis course? Have you noticed any differences about the teaching, the tasks, the textbook?
- Do you work differently to try to understand the subject matter?

## Questions for students, 2nd set

### The exam

- What do you think about the exam?
- How did you experience the five days?
- Do you think that there is a correlation between your score and your own perception of how you understand the subject matter?
- How did you prepare for the exam?
- Have you gained some new insights by trying to solve the four task in the constructed solving sessions?

### The change from the textbook to the notes

- How do you feel about the measure and integration notes?
- Have you missed having the proofs for all the theorems?
- Do you feel that the notes are on a different level than the textbook?
- Did you feel that the teaching changed when you switched from the textbook to the notes?
- Was it an advantage that you worked differently with task solving?

### Study habits

- Did your preparation habits change when you switched from the textbook to the notes?
- Did you read the notes before a lesson?
- Did you try to solve the assigned tasks before a lesson? If you tried but did not succeed what was the reason?
- Do you feel an advantage by reading/trying to work out the proofs by yourself, even though the professor goes through them and you could understand his presentation?

### Teaching and participation

- Have you gained something significant from the dialogues in the lectures? Have there been too little dialogue? If yes, who could have done something about that?
- How do you view your participation in the lectures?

## G.2 The main study

### Questions for the professor

#### The course plan

- Can you describe the plan you have for conducting the course?
- Have you used some of the experiences from conducting the last course?

#### Declaration of the course

- At the first lesson you told the students how you read a mathematics text? What do you hope to achieve by that?
- You have not talked to the students about the role of attending class – in relation to learning the subject matter.



**The course until now**

- How did you experience the progress of the course until now?
- The task solving is displaced compared to the presentation of the chapter. What is the reason for that and are you content with the situation?
- You told me that you were discontent about starting out with chapter one in the textbook. Can you elaborate on the reason for this?

**Questions for students****Conceptions about mathematical analysis and the learning of mathematics**

- What thoughts did you have about learning mathematical analysis before you took the course? Do you think that mathematical analysis and thus learning of mathematical analysis is different from other mathematical topics that you have encountered?
- Does the course fulfil these expectations?
- What role does problem solving play in your attempts to understand the subject matter?
- Do you feel that you need to come up with good ideas or be inventive when you solve tasks?
- Do you need the professor to provide more examples of how to write down correctly the mathematics?

**The course**

- Does your primary learning take place when you attend lessons?
- In the solving sessions you have to work on the tasks from the chapter previously to the one the professor has just reviewed in the lectures. Is that okay with you?

**Preparation and preparation log**

- How do you interpret the categories in the preparation log?
- Do you find them covering compared to the way you prepare?
- What would be the optimal way to prepare for a lesson?
- If you had unlimited amount of time to study would you then attend lectures?

## H Preparation log

**Preparation log**

Name: \_\_\_\_\_ Date for lesson: \_\_\_\_\_

**Acquisition of textbook content:**

Browse through the text/ form preliminary impression of the text	Work through the text with the intention of understanding each argument/ deductive step	Form overview/ think about what is major and minor in the text	Repetition of old material (e.g. from previous chapters, material from previously taken courses)
Time consumption	Time consumption	Time consumption	Time consumption

**Task solving**

Note time consumption, number of tasks (sub-tasks), and if you have worked alone (a) or in groups (g).

Browse through tasks/ consider different strategies			Solution of tasks where you think you have completed the task			Solution of tasks where you give up reaching a solution			Solution of tasks where you decide to terminate the solving process (e.g. due to lack of time)		
Time consumption	Number of tasks (sub-tasks)	alone (a) / group (g)	Time consumption	Number of tasks (sub-tasks)	alone (a) / group (g)	Time consumption	Number of tasks (sub-tasks)	alone (a) / group (g)	Time consumption	Number of tasks (sub-tasks)	alone (a) / group (g)

**Comments:**